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# The formulation and analysis of the nine-point finite difference approximation for the neutron diffusion equation in cylindrical geometry

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# The formulation and analysis of the nine-point finite difference approximation for the neutron diffusion equation in cylindrical geometry

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# NOMENCLATURE

| | | |
|---|---|---|
| $b$ | = | reflector thickness |
| $c$ | = | symbol for the core |
| $e$ | = | symbol for the reflector |
| $i$ | = | subscript for radial coordinate ($i = 1,2,3,4,\ldots..$) |
| $j$ | = | subscript for axial coordinate mesh points ($j = 1, 2,3,4\ldots...$) |
| $r$ | = | radial coordinate |
| $z$ | = | axial coordinate |
| $B$ | = | $h_{z_{j-1}} = z_j - z_{j-1}$, lower axial increment |
| $D(r,z)$ | = | diffusion coefficient |
| $H$ | = | core height |
| $L$ | = | $h_{r_{i-1}} = r_i - r_{i-1}$, left radial increment |
| $M$ | = | $r_i^4 - (r_i + R/2)^4$ |
| $P$ | = | $r_i^2 - (r_i + R/2)^2$ |
| $Q$ | = | $r_i^3 - (r_i + r/2)^3$ |
| $R$ | = | $h_{r_i} = r_{i+1} - r_i$, right radial increment |
| $T$ | = | $h_{z_j} = z_{j+1} - z_j$, top axial increment |
| $B_c$ | = | material buckling |
| $\tilde{H}$ | = | $\frac{H}{2} + b$ |

$I_o$ = modified Bessel's function of the first kind of zero order

$J_o$ = Bessel's function of zero order

$K_o$ = modified Bessel's function of the second kind of zero order

$I_1$ = modified Bessel's function of the first kind of first order

$K_1$ = modified Bessel's function of the second kind of first order

$L_e$ = diffusion length in the reflector

$M'$ = $(r_i - L/2)^4 - r_i^4$

$P'$ = $(r_i - L/2)^2 - r_i^2$

$Q'$ = $(r_i - L/2)^3 - r_i^3$

$R'$ = core radius

$\alpha$ = $(2.405/R')^{1/2}$

$\nu$ = average number of neutrons released per fission

$\Sigma(r,z)$ = $\Sigma_a^{(e)} - \nu\Sigma_f$

$\phi(r,z)$ = neutron flux

$\beta_e$ = $\left[(\Sigma_a^{(e)}/D_e) + \alpha^2\right]^{1/2}$

$\Sigma_a(r,z)$ = macroscopic absorption cross section

$\Sigma_f(r,z)$ = macroscopic fission cross section

$\Sigma_a^{(c)}$ = macroscopic absorption cross section in the core

$\Sigma_a^{(e)}$ = macroscopic absorption cross section in the reflector

$\phi_c(r,z)$ = neutron flux in the core

$\phi_e(r,z)$ = neutron flux in the reflector

# I. INTRODUCTION AND LITERATURE REVIEW

In most practical cases, it is difficult to find an analytical solution to the mathematical problems encountered with reactor physics calculations. In such cases, it is necessary to satisfy appropriate boundary conditions on the boundary of a specified region and to match the continuity requirements within that region. Analytical methods work in certain special cases but are of no practical use in many cases. However, many devised numerical methods have been developed to deal with such cases, especially for the complex reactor configurations where there is almost a continuously varying distribution of materials in the reactor.

In one group theory, the static neutron diffusion equation in two spatial dimensions for a critical reactor may be written [1] as

$$\nabla\cdot D(r,z)\nabla\phi(r,z) - \Sigma(r,z)\phi(r,z) = 0$$

where $\Sigma(r,z) = \Sigma_a(r,z) - \nu\,\Sigma_f(r,z)$

The above equation may be solved approximately by finite-difference methods which could be obtained using a Taylor's expansion method or a variational method [2].

In this numerical approach, the technique is to replace the continuum model by a discrete model, i.e., to replace the linear partial differential equation by an algebraic system of simultaneous linear difference equations. In a typical practical problem these may require a set of 2000, or possibly more, simultaneous linear equations which must be solved using a large digital computer [3].

The five-point [2] and the nine-point [4,5,6,7] finite difference equations for the static neutron diffusion equation have been formulated in xy-geometry. Also, the five-point finite difference equations for the time independent neutron diffusion equation have been formulated in r-z cylindrical geometry for equal and nonequal spacing [3,1,8].

In the present work, the nine-point finite difference equations for the static neutron diffusion equation, considering unequal spacing in r-z cylindrical geometry, have been devised. The nine-point grid of Fig. 1 has been chosen [ 6 ] in order to expand the neutron flux at the corners about the point $(r_i, z_i)$. Another nine-point grid is possible but it requires one to impose additional continuity conditions to meet the cases where some of the derivatives are not continuous across interfaces. However, important aspects to consider in the formulation of the nine-point technique is to insure higher accuracy for the solution of the finite difference equations.

The strategy of solving the finite difference equations depends specifically on the iterative scheme. For elliptic systems, the iterative schemes may be categorized as stationary and nonstationary iterations. However, the main predominant methods in solving elliptic difference equations are successive overrelaxation (SOR) iterative techniques and alternating-direction-implicit (ADI) iterative techniques [9]. In this work, successive overrelaxation has been used without any attempt to optimize the SOR parameter.

# II. THEORY

## A. Neutron Diffusion Equation

The basic steady state neutron diffusion equation in a nonhomogeneous medium is:

$$-\nabla\cdot D(r,z)\nabla\phi(r,z) + \Sigma(r,z)\phi(r,z) - S(r,z) = 0 \qquad (1)$$

The leakage term can be expressed in cylindrical geometry, r-z coordinates, as

$$-\nabla\cdot D\nabla\phi = -\frac{1}{r}\frac{\partial}{\partial r}\left(D\cdot r\frac{\partial\phi}{\partial r}\right) - \frac{\partial}{\partial z}\left(D\frac{\partial\phi}{\partial z}\right) \qquad (2)$$

Insertion of relation (2) into Eq. 1, results in

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(D\cdot r\frac{\partial}{\partial r}\right) - \frac{\partial\phi}{\partial z}\left(D\frac{\partial\phi}{\partial z}\right) + \Sigma\phi - S = 0$$

In the absence of extraneous source, the equation after multiplying both sides by r may be written as

$$\frac{\partial}{\partial r}\left(D\cdot r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(D\cdot r\frac{\partial\phi}{\partial z}\right) - r\Sigma\phi = 0 \qquad (3)$$

In this case any fissioning occurring in the medium has to be contained within the term $\Sigma\phi$ of Eq. 3. Thus, Eq. 3 is a homogeneous equation.

## B. Formulation of Finite Difference Equations

The one-group neutron diffusion equation in two spatial variables is considered over region R in a physical system such as a nuclear reactor. This region contains a homogeneous finite-connected number of sub-regions, $R_n$, (n = 1,2,3,4), as shown in Fig. 1. The increments and the coordinates of the points in Fig. 1 may be written as follows:

1. Increments O(h):

$$r_{i+1} - r_i = h_{r_i} = R \quad \text{Right}$$

$$r_i - r_{i-1} = h_{r_{i-1}} = L \quad \text{Left}$$

$$z_{j+1} - z_j = h_{z_j} = T \quad \text{Top}$$

$$z_j - z_{j-1} = h_{z_{j-1}} = B \quad \text{Bottom}$$

2. Coordinates:

$$o \equiv [r_i \, , \, z_j]$$

$$a \equiv [r_i + \frac{R}{2} \, , \, z_j]$$

$$b \equiv [r_i + \frac{R}{2} \, , \, z_j + \frac{T}{2}]$$

$$c \equiv [r_i \, , \, z_j + \frac{T}{2}]$$

$$e \equiv [r_i - \frac{L}{2} \, , \, z_j + \frac{T}{2}]$$

$$f \equiv [r_i - \frac{L}{2} \, , \, z_j]$$

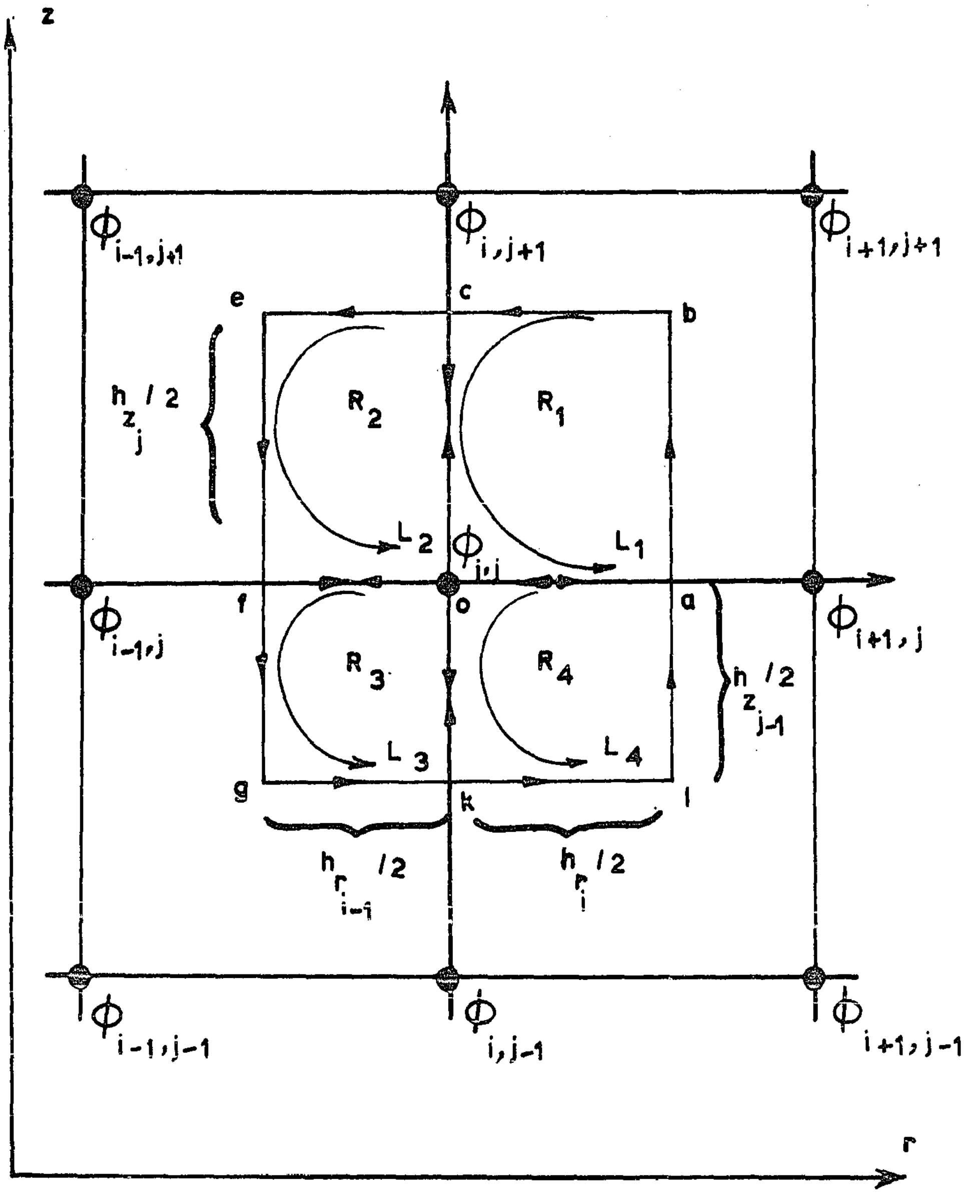


Fig. 1. Mesh structure in the interior

$$g \equiv [r_i - \frac{L}{2}\ ,\ z_j - \frac{B}{2}]$$

$$k \equiv [r_i\ ,\ z_j - \frac{B}{2}]$$

$$\ell \equiv [r_i + \frac{R}{2}\ ,\ z_j - \frac{B}{2}]$$

The region parameters $D_n$, $\Sigma_{an}$, and $\Sigma_{fn}$ are material property parameters which are nonnegative and piece-wise continuous in sub-region, $R_n$. Discontinuities do exist at interfaces between dissimilar media. However, the parameters are usually constants within the region R and the neutron flux and its derivatives are continuous over the region, R, [2]. Thus, these assumptions allow one to perform the surface integral of Eq. 3 for the region R, over the volume of a reactor generated by rotating the rectangular region R about the z-axis ($2\pi rdrdz$)

$$\iint_{R_n} \left[ \frac{\partial}{\partial r}\left( D_n \cdot r \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial z}\left( D_n \cdot r \frac{\partial \phi}{\partial z} \right) - r\Sigma_n \phi(r,z) \right] drdz = 0 \qquad (4)$$

Consequently, the previous assumptions permit the application of Green's theorem in order to transfer the surface integral into a line integral around the boundary $L_n$ of the sub-region $R_n$ oriented in the direction as shown in Fig. 1. Hence, Eq. 4 may be written as

$$\sum_{n=1}^{4} D_n \iint_{L_n} \left[ \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \frac{\partial \phi}{\partial z}\right) \right] drdz - \sum_{n=1}^{4} \iint_{R_n} \Sigma_n \, r\phi(r,z) drdz = 0 \tag{5}$$

C. Application of Green's Theorem on the Leakage Term

Green's theorem can be applied in order to convert the leakage term of Eq. 5 into a line integral. Green's theorem may be expressed as [10]

$$\iint_{R_n} \left(\frac{\partial N}{\partial r} - \frac{\partial M}{\partial z}\right) drdz = \int_{L_n} [M(r,z)dr + N(r,z)dz]$$

where

$$N(r,z) = r \frac{\partial \phi(r,z)}{\partial r}$$

and

$$M(r,z) = -r \frac{\partial \phi(r,z)}{\partial z}$$

Thus, the leakage term of Eq. 5 may be written as

$$\sum_{n=1}^{4} D_n \iint_{R_n} \left(\frac{\partial N}{\partial r} - \frac{\partial M}{\partial z}\right) drdz = \sum_{n=1}^{4} D_n \int_{L_n} [Mdr + Ndz] \tag{7}$$

The right hand side of Eq. 7 may be expanded as

$$\sum_{n=1}^{4} D_n \int_{L_n} [Mdr + Ndz] = D_1 \int_{L_1} \left[-r \frac{\partial \phi}{\partial z} dr + r \frac{\partial \phi}{\partial r} dz\right]$$

$$+ D_2 \int_{L_2} \left[-r \frac{\partial \phi}{\partial z} dr + r \frac{\partial \phi}{\partial r} dz\right] + D_3 \int_{L_3} \left[-r \frac{\partial \phi}{\partial z} dr + r \frac{\partial \phi}{\partial r} dz\right]$$

$$+ D_4 \int_{L_4} \left[-r \frac{\partial \phi}{\partial z} + r \frac{\partial \phi}{\partial r} dz\right] \quad (8)$$

D. Evaluation of the Line Integrals

The line integrals in Eq. 8 can be evaluated using the following techniques:

1. The neutron flux, $\phi(r,z)$, has continuous partial derivatives in a neighborhood $R_n$ for the point $(r_i,z_i)$ in the $(r,z)$ plane. Thus the neutron flux can be expanded about the point $(r_i,z_j)$ using a Taylor's expansion for functions of two variables [8]

$$\phi(r,z) = \sum_{m=o}^{3} \frac{1}{m!} [(r-r_i)D_r + (z-z_j)D_z]^m \phi(r_i,z_j)$$

$$\phi(r,z) = \phi_o + (r-r_i)D_r\phi_o + (z-z_j)D_z\phi_o + \frac{1}{2}(r-r_i)^2 D_r^2\phi_o$$

$$+(r-r_i)(z-z_j)D_{rz}\phi_o + \frac{1}{2}(z-z_j)^2 D_z^2\phi_o + \frac{1}{6}(r-r_i)^3 D_r^3\phi_o$$

$$+ \frac{1}{2}(r-r_i)^2(z-z_j)D_{rrz}\phi_o + \frac{1}{6}(z-z_j)^3 D_z^3\phi_o + O(h^4)$$

$$(9)$$

where

$$\phi_{i,j} = \phi(r_i,z_j) = \phi_o,\ D_r = \frac{\partial}{\partial r}\ ,\ D_z = \frac{\partial}{\partial z}\ ,\ D_{rz} = \frac{\partial^2}{\partial r \partial z}\ ,$$

$$D_{rrz} = \frac{\partial^3}{\partial r \partial r \partial z}\ ,\ \text{and}\ D_{rzz} = \frac{\partial^3}{\partial r \partial z \partial z}$$

2. The partial derivatives, $\frac{\partial \phi(r,z)}{\partial r}$ and $\frac{\partial \phi(r,z)}{\partial z}$, can be obtained by differentiating the flux $\phi(r,z)$ in Eq. 9 with respect to r and z, respectively, that is

$$\frac{\partial \phi(r,z)}{\partial r} = D_r\phi_o + (r-r_i)D_r^2\phi_o + (z-z_j)D_{rz}\phi_o + \frac{1}{2}(r-r_i)(z-z_j)D_{rrz}\phi_o + \frac{1}{2}(z-z_j)^2D_{rzz}\phi_o + O(h^3) \qquad (10)$$

and

$$\frac{\partial \phi(r,z)}{\partial z} = D_z\phi_o + (r-r_i)D_{rz}\phi_o + (z-z_j)D_z^2\phi_o + \frac{1}{2}(r-r_i)^2D_{rrz}\phi_o + \frac{1}{2}(r-r_i)(z-z_j)D_{rzz}\phi_o + \frac{1}{2}(z-z_j)^2\ D_z^3\phi_o + O(h^3) \qquad (11)$$

Therefore, $\frac{\partial \phi(r,z)}{\partial r}$ and $\frac{\partial \phi(r,z)}{\partial z}$, can be evaluated at the points $(r_i + \frac{R}{2},\ z-z_j)$, $(r_i - \frac{L}{2},\ z-z_j)$, $(r-r_i,\ z + \frac{T}{2})$, and $(r-r_i,\ z_j - \frac{B}{2})$. The results are

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i+\frac{R}{2}} = D_r\phi_o + \frac{R}{2} D_r^2\phi_o + (z-z_j)D_{rz}\phi_o + \frac{R^2}{8} D_r^3\phi_o + \frac{R}{2} (z-z_j)D_{rrz}\phi_o + \frac{1}{2} (z-z_j)^2 D_{rzz}\phi_o \tag{12}$$

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i-\frac{1}{2}} = D_r\phi_o - \frac{L}{2} D_r^2\phi_o + (z-z_j)D_{rz}\phi_o + \frac{L^2}{8} D_r^3\phi_o - \frac{L}{2} (z-z_j)D_{rrz}\phi_o + \frac{1}{2} (z-z_j)^2 D_{rzz}\phi_o \tag{13}$$

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=z_j+\frac{T}{2}} = D_z\phi_o + (r-r_i)D_{rz}\phi_o + \frac{T}{2} D_z^2\phi_o + \frac{1}{2} (r-r_i)^2 D_{rrz}\phi_o + \frac{T}{2} (r-r_i) D_{rzz}\phi_o + \frac{T^2}{8} D_z^3\phi_o \tag{14}$$

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=z_j-\frac{B}{2}} = D_z\phi_o + (r-r_i)D_{rz}\phi_o - \frac{B}{2} D_z^2\phi_o + \frac{1}{2} (r-r_i)^2 D_{rrz}\phi_o - \frac{B}{2} (r-r_i) D_{rzz}\phi_o + \frac{B^2}{8} D_z^3\phi_o \tag{15}$$

3. The continuity of the neutron current density can be applied at material interfaces in order to eliminate the forward and backward line integrals along the same paths. This may be visualized as follows:

$$- D_1 \int_o^a r\left(\frac{\partial\phi}{\partial z}\right)_{R_1} dr - D_4 \int_a^o r\left(\frac{\partial\phi}{\partial z}\right)_{R_4} dr$$

$$= - D_1 \int_o^a r\left(\frac{\partial\phi}{\partial z}\right)_{R_1} dr + D_4 \int_o^a r\left(\frac{\partial\phi}{\partial z}\right)_{R_4} dr$$

But $D_1\left(\frac{\partial\phi}{\partial z}\right)_{R_1} = D_4\left(\frac{\partial\phi}{\partial z}\right)_{R_4}$ because the neutron current density is a continuous function across material interfaces. Thus,

$\int r[-D_1\left(\frac{\partial\phi}{\partial z}\right)_{R_1} dr + D_4\left(\frac{\partial\phi}{\partial z}\right)_{R_4} dr] = 0$. Similar relations occur along (o,c), (o,f), and o,k).

Therefore, the line integrals in Eq. 8 may be evaluated by performing the integration along the paths $L_1$, $L_2$, $L_3$, and $L_4$ as follows:

a:

$$D_1 \int_{L_1} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right] = -D_1\int_b^c r\left(\frac{\partial\phi}{\partial z}\right)_{z=z_j+\frac{T}{2}} dr$$

$$+ D_1 \int_a^b r\left(\frac{\partial\phi}{\partial r}\right)_{r=r_i+\frac{R}{2}} dz$$

In Appendix B are developed the expressions for these various partial derivatives. These expressions are then further algebraically simplified. By substituting for $\left(\frac{\partial\phi}{\partial z}\right)_{z=z_j+\frac{T}{2}}$ and

$\left(\frac{\partial\phi}{\partial r}\right)_{r=r_i+\frac{R}{2}}$ from Eq. 11 and Eq. 4 of Appendix B, one has

$$D_1\int_{L_1}\left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right] = -D_1\int_{r_i+\frac{R}{2}}^{r_1} r[G + (r-r_i)H + (r-r_i)^2K]dr$$

$$+D_1\int_{z_j}^{z_j+\frac{T}{2}} r[A + (z-z_j)F + (z-z_j)^2C]dz$$

$$=-\frac{1}{2}D_1G[r_i^2-(r_i+\frac{R}{2})^2]-D_1H\cdot\{\frac{1}{3}[r_i^3-(r_i+\frac{R}{2})^3-\frac{1}{2}r_i[r_i^2-(r_i+\frac{R}{2})^2]\}$$

$$-D_1K\cdot\{\frac{1}{4}[r_i^4-(r_i+\frac{R}{2})^4]-\frac{2}{3}r_i[r_i^3-(r_i+\frac{R}{2})^3]+\frac{1}{2}r_i^2[r_i^2-(r_i+\frac{R}{2})^2]\}$$

$$+D_1A(r_i+\frac{R}{2})\cdot\frac{T}{2} + D_1F(r_i+\frac{R}{2})\cdot\frac{T^2}{8} + D_1(r_i+\frac{R}{2})C\cdot\frac{T^3}{24} \tag{16}$$

Define:

$$P = r_i^2 - (r_i + \frac{R}{2})^2$$

$$Q = r_i^3 - (r_i + \frac{R}{2})^3$$

$$M = r_i^4 - (r_i + \frac{R}{2})^4$$

By substituting for G, H, K, A, F, and C in Eq. 16 from Eqs. 8, 9, 1, 2', and 3 of Appendix B and collecting all the terms of orders $O(h^4)$ and higher under the truncation error $O(h^4)$, one has

$$D_1 \int_{L_1} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right]$$

$$= -\frac{1}{2}\frac{D_1 P}{T}(\phi_2 - \phi_o)$$

$$- \frac{D_1}{R}\left[\frac{1}{T}\left(\frac{1}{3}Q - \frac{1}{2}r_i P\right) - \frac{T}{8}\left(r_i + \frac{R}{2}\right)\right](\phi^1 - \phi_1 - \phi_2 + \phi_o)$$

$$+ \frac{1}{2}\frac{D_1 T}{R}\left(r_i + \frac{R}{2}\right)(\phi_1 - \phi_o)$$

$$+ \frac{1}{2}D_1\left[R\left(\frac{1}{3}Q - \frac{1}{2}r_i P\right) - \left(\frac{1}{4}M - \frac{2}{3}r_i Q + \frac{1}{2}r_i^2 P\right)\right]D_{rrz}\phi_o$$

$$+ \frac{1}{48}D_1 T^2 P D_z^3\phi_o$$

$$- \frac{1}{48}D_1 r_i R^2 T D_r^3\phi_o$$

$$- \frac{1}{24}D_1 r_i T^3 D_{rzz}\phi_o \qquad (17)$$

b:

$$D_2 \int_{L_2} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right]$$

$$= -D_2 \int_c^e r\left(\frac{\partial\phi}{\partial z}\right)_{z=z_j+\frac{T}{2}} dr + D_2 \int_e^f r\left(\frac{\partial\phi}{\partial r}\right)_{r=r_i-\frac{L}{2}} dz$$

By substituting for $\left(\frac{\partial\phi}{\partial z}\right)_{z=z_j+\frac{T}{2}}$ and $\left(\frac{\partial\phi}{\partial r}\right)_{r=r_i-\frac{L}{2}}$ from Eq. 11 and Eq. 7 of Appendix B, the result is

$$D_2 \int_{L_2} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right]$$

$$= -D_2 \int_{r_i}^{r_i-\frac{L}{2}} r[G + (r-r_i)H + (r-r_i)^2K]dr$$

$$+D_2 \int_{z_j+\frac{T}{2}}^{z_j} r[A' + (z-z_j)F' + (z-z_j)^2C]dz \qquad (18)$$

Define:

$$P' = (r_i - \frac{L}{2})^2 - r_i^2$$

$$Q' = (r_i - \frac{L}{2})^3 - r_i^3$$

$$M' = (r_i - \frac{L}{2})^4 - r_i^4$$

By substituting for G, H, K, A', F', and C in Eq. 18 from Eqs. 8, 9', 10, 5, 6, and 3 of Appendix A and collecting all the terms of orders $O(h^4)$ and higher under the truncation error $O(h^4)$, one has

$$D_2 \int_{L_2} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right]$$

$$= -\frac{1}{2}\frac{D_2P'}{T} \quad (\phi_2 - \phi_o)$$

$$+ \frac{D_2}{L}\left[\frac{1}{T}\left(\frac{1}{3}\,Q' - \frac{1}{2}\,r_iP'\right) + \frac{T}{8}\left(r_i - \frac{L}{2}\right)\right](\phi^2 - \phi_3 - \phi_2 + \phi_o)$$

$$+ \frac{1}{2}\frac{D_2T}{L}\left(r_i - \frac{L}{2}\right) \; (\phi_3 - \phi_o)$$

$$+ \frac{1}{48}\, D_2TL^2r_iD_r^3\phi_o$$

$$- \frac{1}{2}\, D_2\left[\left(\frac{1}{4}M' - \frac{2}{3}\,r_iQ' + \frac{1}{2}\,r_i^2P'\right)\right.$$

$$\left. + L\,\frac{1}{3}\,Q' - \frac{1}{2}\,r_iP'\right)\Big] D_{rrz}\phi_o$$

$$+ \frac{1}{24}\, D_2T^3r_iD_{rzz}\phi_o \qquad (19)$$

c:

$$D_3 \int_{L_3} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right] = -D_3 \int_g^k r\left(\frac{\partial\phi}{\partial z}\right)_{z=z_j-\frac{B}{2}} dr$$

$$+ D_3 \int_f^g r\left(\frac{\partial\phi}{\partial r}\right)_{r=r_i-\frac{L}{2}} dz$$

By substituting for $\left(\frac{\partial\phi}{\partial z}\right)_{z=z_j-\frac{B}{2}}$ and $\left(\frac{\partial\phi}{\partial r}\right)_{r=r_i-\frac{L}{2}}$ from Eq. 4 and Eq. 7 of Appendix B, one has

$$D_3 \int_{L_3} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right] = -D_3 \int_{r_i-\frac{L}{2}}^{r_i} r[G' + (r-r_i)H' + (r-r_i)^2K]dr$$

$$+D_3 \int_{z_j}^{z_j-\frac{B}{2}} r[A' + (z-z_j)F' + (z-z_j)^2C]dz \qquad (20)$$

By substituting for G', H', K, A', F', and C in Eq. 20 from the Eqs. 12, 13, 10, 5, 6', and 3 of Appendix B and collecting all the terms of orders $O(h^4)$ and higher under the truncation error $O(h^4)$ one has

$$D_3 \int_{L_1} \left[-r\left(\frac{\partial\phi}{\partial z}\right)dr + r\left(\frac{\partial\phi}{\partial r}\right)dz\right]$$

$$= -\frac{1}{2}\frac{D_3P'}{B}(\phi_4 - \phi_o)$$

$$+ \frac{D_3}{L}\left[\frac{1}{B}\left(\frac{1}{3}Q' - \frac{1}{2}r_iP'\right) + \frac{B}{8}\left(r_i - \frac{L}{2}\right)\right](\phi^3-\phi_3-\phi_4+\phi_o)$$

$$+ \frac{1}{2}\frac{D_3B}{L}\left(r_i - \frac{L}{2}\right)(\phi_3 - \phi_o)$$

$$- \frac{1}{48} D_3 B^2 P' D_z^3 \phi_o$$

$$+ \frac{1}{48} D_3 r_i L^3 B D_r^3 \phi_o$$

$$+ \frac{1}{2} D_3 \left[ \left( \frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i^2 P' \right) + L \left( \frac{1}{3} Q' - \frac{1}{2} r_i P' \right) \right] D_{rrz} \phi_o$$

$$+ \frac{1}{24} D_3 B^3 r_i D_{rzz} \phi_o \qquad (21)$$

d:

$$D_4 \int_{L_4} \left[ -r\left(\frac{\partial \phi}{\partial z}\right) dr + r\left(\frac{\partial \phi}{\partial r}\right) dz \right]$$

$$= - D_4 \int_k^{\ell} r \left(\frac{\partial \phi}{\partial z}\right)_{z=z_j-\frac{B}{2}} dr + D_4 \int_{\ell}^{a} r \left(\frac{\partial \phi}{\partial r}\right)_{r=r_i+\frac{R}{2}} dz$$

By substituting for $\left(\frac{\partial \phi}{\partial z}\right)_{z=z_j-\frac{B}{2}}$ and $\left(\frac{\partial \phi}{\partial r}\right)_{r=r_i+\frac{R}{2}}$ from Eq. 14 and Eq. 4 of Appendix B one has

$$D_4 \int_{L_4} \left[ -r\left(\frac{\partial \phi}{\partial z}\right) dr + r\left(\frac{\partial \phi}{\partial r}\right) dz \right] = -D_4 \int_{r_i}^{r_i+\frac{R}{2}} r[G' + (r-r_i)H' + (r-r_i)^2 K] dr$$

$$+ D_4 \int_{z_j-\frac{B}{2}}^{z_j} r[A + (z-z_j)F + (z-z_j)^2 C] dz \qquad (22)$$

By substituting for G', H', K, A, F, and C in Eq. 22 from the Eqs. 12, 13', 10, 1, 2, and 3 of Appendix B and collecting all the terms of orders $O(h^4)$ and higher under the truncation error $O(h^4)$ one has

$$D_4 \int_{L_4} \left[ -r\left(\frac{\partial \phi}{\partial z}\right) dr + r\left(\frac{\partial \phi}{\partial r}\right) dz \right]$$

$$= -\frac{1}{2}\frac{D_4 P}{B}(\phi_4 - \phi_o)$$

$$- \frac{D_4}{R}\left[\frac{1}{B}\left(\frac{1}{3} Q - \frac{1}{2} r_i P\right) - \frac{B}{8}\left(r_i + \frac{R}{2}\right)\right](\phi^4 - \phi_1 - \phi_4 + \phi_o)$$

$$+ \frac{1}{2}\frac{D_4 B}{R}\left(r_i + \frac{R}{2}\right)(\phi_1 - \phi_o)$$

$$- \frac{1}{48} D_4 B^2 P D_z^3 \phi_o$$

$$- \frac{1}{48} D_4 B R^2 r_p D_r^3 \phi_o$$

$$+ \frac{1}{2} D_4 \left[\left(\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P\right) - R\left(\frac{1}{3} Q - \frac{1}{2} r_i P\right)\right] D_{rrz}\phi_o$$

$$- \frac{1}{2} D_4 r_i B^3 D_{rzz}\phi_o \qquad (23)$$

By substituting for the terms that contain the derivatives in the Eqs. 17, 19, 21, and 23 from the relations 25, 33, 36, and 39 of Appendix B one has

$$\sum_{n=1}^{4} D_n \iint_{R_n} \left[\frac{\partial}{\partial r}\left(r\,\frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\,\frac{\partial \phi}{\partial z}\right)\right] drdz$$

$$= -\frac{1}{2}\,\frac{D_1 P}{T}\,(\phi_2 - \phi_o)$$

$$-\frac{D_1}{R}\left[\frac{1}{T}\left(\frac{1}{3}\,Q - \frac{1}{2}\,r_i P\right) - \frac{T}{8}\left(r_i + \frac{R}{2}\right)\right](\phi^1 - \phi_1 - \phi_2 + \phi_o)$$

$$+\frac{1}{2}\,\frac{D_1 T}{R}\left(r_i + \frac{R}{2}\right)\,(\phi_1 - \phi_o)$$

$$-\frac{1}{48}\,D_1 r_i RT(TD_z^3\phi_o + RD_r^3\phi_o)\Big|_{R_1}$$

$$-\frac{1}{24}\,D_1 r_i(R^3 D_{rrz}\phi_o + T^3\,D_{rzz}\phi_o)\Big|_{R_1}$$

$$-\frac{1}{2}\,\frac{D_2 P'}{T}\,(\phi_2 - \phi_o)$$

$$+\frac{D_2}{L}\left[\frac{1}{T}\left(\frac{1}{3}\,Q' - \frac{1}{2}\,r_i P'\right) + \frac{T}{8}\left(r_i - \frac{L}{2}\right)\right](\phi^2 - \phi_3 - \phi_2 + \phi_o)$$

$$+\frac{1}{2}\,\frac{D_2 T}{L}\left(r_i - \frac{L}{2}\right)\,(\phi_3 - \phi_o)$$

$$-\frac{1}{48}\,D_2 r_i LT(TD_z^3\ {}_o - LD_r^3\phi_o\ \Big|_{R_2}$$

$$-\frac{1}{24}\,D_2 r_i(L^3\,D_{rrz}\phi_o - T^3\,D_{rzz}\phi_o)\Big|_{R_2}$$

$$- \frac{1}{2} \frac{D_3 P'}{B} (\phi_4 - \phi_o)$$

$$+ \frac{D_3}{L} \left[ \frac{1}{B} \left( \frac{1}{3} Q' - \frac{1}{2} r_i P' \right) + \frac{B}{8} \left( r_i - \frac{L}{2} \right) \right] (\phi^3 - \phi_3 - \phi_4 + \phi_o)$$

$$+ \frac{1}{2} \frac{D_3 B}{L} \left( r_i - \frac{L}{2} \right) (\phi_3 - \phi_o)$$

$$+ \frac{1}{48} D_3 \ r_i \ BL(BD_z^3 \phi_o + LD_r^3 \phi_o) \Big|_{R_3}$$

$$+ \frac{1}{24} D_3 r_i (L^3 \ D_{rrz} \phi_o + B^3 \ D_{rzz} \phi_o) \Big|_{R_3}$$

$$- \frac{1}{2} \frac{D_4 P}{B} (\phi_4 - \phi_o)$$

$$- \frac{D_4}{R} \left[ \frac{1}{B} \left( \frac{1}{3} Q - \frac{1}{2} r_i P \right) - \frac{B}{8} \left( r_i + \frac{R}{2} \right) \right] (\phi^4 - \phi_1 - \phi_4 + \phi_o)$$

$$+ \frac{1}{2} \ \frac{D_4 B}{R} \left( r_i + \frac{R}{2} \right) (\phi_1 - \phi_o)$$

$$+ \frac{1}{48} \ D_4 BRr_i (BD_z^3 \ {}_o - RD_r^3 \ {}_o) \quad {}_{R_4}$$

$$+ \frac{1}{24} \ D_4 r_i (R^3 D_{rrz} \phi_o - B^3 D_{rzz} \phi_o) \Big|_{R_4} \qquad (24)$$

In Appendix C, the identity $\nabla^2 \phi = \frac{\Sigma_n}{D_n} \ \phi$ and a Taylor's expansion are applied to perform further simplification on the derivative terms of Eq. 24. By substituting for these terms in Eq. 24 from Eq. 31 of Appendix C, one has

$$\sum_{n=1}^{4} D_n \iint_{R_n} \left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\phi}{\partial z}\right)\right] drdz$$

$$= -\frac{1}{2}\frac{D_1 P}{T}(\phi_2 - \phi_o)$$

$$-\frac{D_1}{R}\left[\frac{1}{T}\left(\frac{1}{3}Q - \frac{1}{2}r_i P\right) - \frac{T}{8}\left(r_i + \frac{R}{2}\right)\right](\phi^1 - \phi_1 - \phi_2 + \phi_o)$$

$$+\frac{1}{2}\frac{D_1 T}{R}\left(r_i + \frac{R}{2}\right)(\phi_1 - \phi_o)$$

$$-\frac{1}{2}\frac{D_2 P'}{T}(\phi_2 - \phi_o)$$

$$+\frac{D_2}{L}\left[\frac{1}{T}\left(\frac{1}{3}Q' - \frac{1}{2}r_i P'\right) + \frac{T}{8}\left(r_i - \frac{L}{2}\right)\right](\phi^2 - \phi_3 - \phi_2 + \phi_o)$$

$$+\frac{1}{2}\frac{D_2 T}{L}\left(r_i - \frac{L}{2}\right)(\phi_3 - \phi_o)$$

$$-\frac{1}{2}\frac{D_3 P'}{B}(\phi_4 - \phi_o)$$

$$+\frac{D_3}{L}\left[\frac{1}{B}\left(\frac{1}{3}Q' - \frac{1}{2}r_i P'\right) + \frac{B}{8}\left(r_i - \frac{L}{2}\right)\right](\phi^3 - \phi_3 - \phi_4 + \phi_o)$$

$$+\frac{1}{2}\frac{D_3 B}{L}\left(r_i - \frac{L}{2}\right)(\phi_3 - \phi_o)$$

$$-\frac{1}{2}\frac{D_4 P}{B}(\phi_4 - \phi_o)$$

$$-\frac{D_4}{R}\left[\frac{1}{B}\left(\frac{1}{3}Q - \frac{1}{2}r_i P\right) - \frac{B}{8}\left(r_i + \frac{R}{2}\right)\right](\phi^4 - \phi_1 - \phi_4 + \phi_o)$$

$$+ \frac{1}{2} \frac{D_4 B}{R} \left( r_i + \frac{R}{2} \right) (\phi_1 - \phi_o)$$

$$- \frac{1}{48} D_1 \sigma_1 r_i RT (\phi_1 + \phi_2 - 2\phi_o)$$

$$- \frac{1}{48} D_2 \sigma_2 r_i LT (\phi_2 + \phi_3 - 2\phi_o)$$

$$- \frac{1}{48} D_3 \sigma_3 r_i LB (\phi_4 + \phi_3 - 2\phi_o)$$

$$- \frac{1}{48} D_4 \sigma_4 r_i RB (\phi_4 + \phi_1 - 2\phi_o)$$

$$+ \frac{1}{24} \frac{T}{LR} (R - L) [LD_1(\phi_1 - \phi_o) + RD_2(\phi_3 - \phi_o)]$$

$$+ \frac{1}{24} \frac{B}{LR} (R - L) [LD_4(\phi_1 - \phi_o) + RD_3(\phi_3 - \phi_o)]$$

$$+ \frac{1}{46} \frac{T}{LR} (L + R)[LD_1(\phi^1 - \phi_1 - \phi_2 + \phi_o) - RD_2(\phi^2 - \phi_3 - \phi_2 + \phi_o)]$$

$$+ \frac{1}{46} \frac{B}{LR} (L + R)[LD_4(\phi^4 - \phi_1 - \phi_4 + \phi_o) - RD_3(\phi^3 - \phi_3 - \phi_4 + \phi_o)]$$

$$+ \frac{1}{24} \frac{r_i R}{TB} [D_1 B(\phi^1 - \phi_1 - \phi_2 + \phi_o) + D_4 T(\phi^4 - \phi_1 - \phi_4 + \phi_o)]$$

$$+ \frac{1}{24} \frac{r_i L}{TB} [D_2 B(\phi^2 - \phi_3 - \phi_2 + \phi_o) + D_3 T(\phi^3 - \phi_3 - \phi_4 + \phi_o)]$$

$$+ \frac{1}{24} \frac{r_i T}{LR} [D_1 L(\phi^1 - \phi_1 - \phi_2 + \phi_o) + D_2 R(\phi^2 - \phi_3 - \phi_2 + \phi_o)]$$

$$+ \frac{1}{24}\frac{r_i B}{LR}\ [D_3 R(\phi^3-\phi_3-\phi_4+\phi_o)+D_4 L(\phi^4-\phi_1-\phi_4+\phi_o)]$$

$$+ O(h^4) \qquad (25)$$

E. Evaluation of the Surface Integral

Consider the surface integral term

$$-\sum_{n=1}^{4} \iint_{R_n} r\ \Sigma_n\ \phi(r,z)drdz \text{ of Eq. 5.}$$

In this case, the Taylor series for the flux expansion is truncated to $O(h^2)$ since the truncation error will be of order $O(h^4)$ after performing the integration. The flux expansion is

$$\phi(r,z) = \phi_o + (r-r_i)D_r\phi_o + (z-z_j)D_z\phi_o$$

Thus, by inserting this flux expansion into the above integral term, one has

$$-\sum_{n=1}^{4} \iint r\ \Sigma_n\ [\phi_o + (r-r_i)D_r\phi_o + (z-z_j)D_z\phi_o]drdz$$

The integration may be performed as follows:

a: For n = 1

$$-\Sigma_1 \int_{z_j}^{z_j+\frac{T}{2}} \int_{r_i}^{r_i+\frac{R}{2}} r[\phi_o + (r-r_i)D_r\phi_o\Big|_{R_1} + (z-z_j)D_z\phi_1\Big|_{R_1}]\ drdz$$

$$= \frac{1}{4} \Sigma_1 TP\phi_o + \frac{1}{2} \Sigma_1 T\left(\frac{1}{3} Q - \frac{1}{2} r_i P\right) D_r\phi_o\Big|_{R_1} + \frac{1}{16} \Sigma_1 T^2 P D_z \phi_o\Big|_{R_1} \quad (26)$$

b: For n = 2

$$- \Sigma_2 \int_{r_i-\frac{L}{2}}^{r_i} \int_{z_j}^{z_j+\frac{T}{2}} r[\phi_o + (r-r_i) D_r\phi_o\Big|_{R_2} + (z-z_j) D_z\phi_o\Big|_{R_2}]\, drdz$$

$$= \frac{1}{4} \Sigma_2\ T\ P'\ \phi_o + \frac{1}{2} \Sigma_2\ T\left(\frac{1}{3} Q' - \frac{1}{2} r_i P'\right) D_r\phi_o\Big|_{R_2}$$

$$+ \frac{1}{16} \Sigma_2\ T^2\ P'\ D_z\phi_o\Big|_{R_2} \quad (27)$$

c: For n = 3

$$- \Sigma_3 \int_{z_j-\frac{B}{2}}^{z_j} \int_{r_i-\frac{L}{2}}^{r_i} r[\phi_o + (r-r_i) D_r\phi_o\Big|_{R_3} + (z-z_j) D_z\phi_o\Big|_{R_3}]\, drdz$$

$$= \frac{1}{4} \Sigma_3\ BP'\ \phi_o$$

$$+ \frac{1}{2} \Sigma_3 B\left(\frac{1}{3} Q' - \frac{1}{2} r_i P'\right) D_r\phi_o\Big|_{R_3}$$

$$- \frac{1}{16} \Sigma_3 B^2\ P'\ D_z\phi_o\Big|_{R_3} \quad (28)$$

d: For n = 4

$$- \Sigma_4 \int_{z_j-\frac{B}{2}}^{z_j} \int_{r_i}^{r_i+\frac{R}{2}} r[\phi_o + (r-r_i)D_r\phi_o\Big|_{R_4} + (z-z_j)D_z\phi_o\Big|_{R_4}]drdz$$

$$= \frac{1}{4}\Sigma_4 \, B \, P \, \phi_o + \frac{1}{2}\Sigma_4 \, B\left(\frac{1}{3}Q - \frac{1}{2}r_i \, P\right) D_r\phi_o\Big|_{R_4}$$

$$- \frac{1}{16}\Sigma_4 \, B^2 \, P \, D_z\phi_o\Big|_{R_4} \qquad (29)$$

The Eqs. 26, 27, 28, and 29 may be simplified by substituting for P, $\left(\frac{1}{3}Q - \frac{1}{2}r_iP\right)$, P', and $\left(\frac{1}{3}Q' - \frac{1}{2}r_iP'\right)$ from Eqs. 15, 18, 26, and 29 of Appendix B. All the terms of orders $O(h^4)$ and higher are lumped under the truncation error $O(h^4)$. Hence, Eqs. 26, 27, 28, and 29 may be written as

$$- \Sigma_1 \int_{z_j}^{z_j+\frac{T}{2}} \int_{r_i}^{r_i+\frac{R}{2}} r\phi(r,z)drdz = \frac{1}{4}\Sigma_1 \, T \, P \, \phi_o$$

$$- \frac{1}{16}\Sigma_1 \, T \, r_iR^2 \, D_r\phi_o\Big|_{R_1}$$

$$- \frac{1}{16}\Sigma_1 \, T^2 \, r_iR \, D_z\phi_o\Big|_{R_1} \qquad (30)$$

$$-\Sigma_2 \int_{r_i-\frac{L}{2}}^{r_i} \int_{z_j}^{z_j+\frac{T}{2}} r\,\phi(r,z)\,dr dz = \frac{1}{4}\Sigma_2\, T\, P'\, \phi_o + \frac{1}{16}\Sigma_2\, r_i T\, L^2\, D_r\phi_o\Big|_{R_2} - \frac{1}{16}\Sigma_2\, r_i L\, T^2\, D_z\phi_o\Big|_{R_2} \tag{31}$$

$$-\Sigma_3 \int_{z_j-\frac{B}{2}}^{z_j} \int_{r_i-\frac{L}{2}}^{r_i} r\,\phi(r,z)\,dr dz = +\frac{1}{4}\Sigma_3\, B\, P'\, \phi_o + \frac{1}{16}\Sigma_3\, Br_i\, L^2\, D_r\phi_o\Big|_{R_3} + \frac{1}{16}\Sigma_3\, B^2 r_i\, L\, D_z\phi_o\Big|_{R_3} \tag{32}$$

$$-\Sigma_4 \int_{z_j-\frac{B}{2}}^{z_j} \int_{r_i+\frac{R}{2}}^{r_i} r\,\phi(r,z)\,dr dz = +\frac{1}{4}\Sigma_4\, B\, P\, \phi_o - \frac{1}{16}\Sigma_4\, r_i\, BR^2\, D_r\phi_o\Big|_{R_4} + \frac{1}{16}\Sigma_4\, r_i\, B^2 R\, D_z\phi_o\Big|_{R_4} \tag{33}$$

By adding Eqs. 30, 31, 32, and 33, one obtains

$$
\begin{aligned}
-\sum_{n=1}^{4} \iint_{R_n} \Sigma_n \, r\phi(r,z)\,dr\,dz = {} & \frac{1}{4}\Sigma_1 \, T \, P \, \phi_o \\
& - \frac{1}{16}\Sigma_1 \, r_i \, TR^2 \, D_r\phi_o\Big|_{R_1} \\
& - \frac{1}{16}\Sigma_1 \, r_i \, RT^2 \, D_z\phi_o\Big|_{R_1} \\
& + \frac{1}{4}\Sigma_2 \, r_i \, TP' \, \phi_o \\
& + \frac{1}{16}\Sigma_2 \, r_i \, TL^2 \, D_r\phi_o\Big|_{R_2} \\
& - \frac{1}{16}\Sigma_2 \, r_i \, LT^2 \, D_z\phi_o\Big|_{R_2} \\
& + \frac{1}{4}\Sigma_3 \, B \, P' \, \phi_o \\
& + \frac{1}{16}\Sigma_3 \, Br_i \, L^2 \, D_r\phi_o\Big|_{R_3} \\
& + \frac{1}{16}\Sigma_3 \, B^2 r_i \, L \, D_z\phi_o\Big|_{R_3} \\
& + \frac{1}{4}\Sigma_4 \, B \, P \, \phi_o \\
& - \frac{1}{16}\Sigma_4 \, r_i B \, R^2 \, D_r\phi_o\Big|_{R_4} \\
& + \frac{1}{16}\Sigma_4 \, r_i R \, B^2 \, D_z\phi_o\Big|_{R_4}
\end{aligned}
\qquad (34)
$$

Substituting for the terms that contain the derivatives in Eq. 34 from Eq. 41 of Appendix C, one has

$$- \sum_{n=1}^{4} \iint_{R_n} \Sigma_n \, r\phi(r,z)\,drdz = \frac{1}{4}\Sigma_1 \; T \; P \; \phi_o$$

$$+ \frac{1}{4}\Sigma_2 \; T \; P' \; \phi_o$$

$$+ \frac{1}{4}\Sigma_3 \; B \; P' \; \phi_o$$

$$+ \frac{1}{4}\Sigma_4 \; B \; P \; \phi_o$$

$$- \frac{1}{16}\Sigma_1 \; Tr_i R(\phi_1 + \phi_2 - 2\phi_o)$$

$$- \frac{1}{16}\Sigma_2 \; Tr_i L(\phi_3 + \phi_2 - 2\phi_o)$$

$$- \frac{1}{16}\Sigma_3 \; Br_i L(\phi_3 + \phi_4 - 2\phi_o)$$

$$- \frac{1}{16}\Sigma_4 \; Br_i R(\phi_1 + \phi_4 - 2\phi_o)$$

$$+ O(h^4) \qquad (35)$$

Finally Eq. 5 may be expressed in a finite difference form by adding Eq. 25 and Eq. 35. The result is

$$\sum_{n=1}^{4} D_n \iint_{R_n} \left[\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \frac{\partial \phi}{\partial z}\right)\right] drdz - \sum_{n=1}^{4} \iint_{R_n} \Sigma_n r\phi(r,z)\,drdz = 0$$

and

$$- \frac{1}{2} \frac{D_1P}{T} (\phi_2 - \phi_o)$$

$$- \frac{D_1}{R}\left[\frac{1}{T}\left(\frac{1}{3} Q - \frac{1}{2} r_iP\right) - \frac{T}{8} \left(r_i + \frac{R}{2}\right)\right] (\phi^1 - \phi_1 - \phi_2 + \phi_o)$$

$$+ \frac{1}{2} \frac{D_1T}{R} (r_i + \frac{R}{2}) (\phi_1 - \phi_o)$$

$$- \frac{1}{2} \frac{D_2P'}{T} (\phi_2 - \phi_o)$$

$$+ \frac{D_2}{L} \left[\frac{1}{T}\left(\frac{1}{3} Q' - \frac{1}{2} r_iP'\right) + \frac{T}{8} \left(r_i - \frac{L}{2}\right)\right] (\phi^2 - \phi_3 - \phi_2 + \phi_o)$$

$$+ \frac{1}{2} \frac{D_2T}{L} \left(r_i - \frac{L}{2}\right) (\phi_3 - \phi_o)$$

$$- \frac{1}{2} \frac{D_3P'}{B} (\phi_4 - \phi_o)$$

$$+ \frac{D_3}{L} \left[\frac{1}{B}\left(\frac{1}{3} Q' - \frac{1}{2} r_iP'\right) + \frac{B}{8} \left(r_i - \frac{L}{2}\right)\right] (\phi^3 - \phi_3 - \phi_4 + \phi_o)$$

$$+ \frac{1}{2} \frac{D_3B}{L} (\phi_3 - \phi_o) \left(r_i - \frac{L}{2}\right)$$

$$- \frac{1}{2} \frac{D_4P}{B} (\phi_4 - \phi_o)$$

$$- \frac{D_4}{R} \left[\frac{1}{B}\left(\frac{1}{3} Q - \frac{1}{2} r_iP\right) - \frac{B}{8} \left(r_i + \frac{R}{2}\right)\right] (\phi^4 - \phi_1 - \phi_4 + \phi_o)$$

$$+\frac{1}{2}\,\frac{D_4B}{R}\left(r_i + \frac{R}{2}\right)(\phi_1 - \phi_o)$$

$$-\frac{1}{48}\,D_1\;\sigma_1\;r_iR\;T(\phi_1 + \phi_2 - 2\phi_o)$$

$$-\frac{1}{48}\,D_2\;\sigma_2\;r_iL\;T(\phi_2 + \phi_3 - 2\phi_o)$$

$$-\frac{1}{48}\,D_3\;\sigma_3\;r_iL\;B(\phi_4 + \phi_3 - 2\phi_o)$$

$$-\frac{1}{48}\,D_4\;\sigma_4\;r_iR\;B(\phi_4 + \phi_1 - 2\phi_o)$$

$$+\frac{1}{24}\,\frac{T}{LR}(R - L)[LD_1(\phi_1 - \phi_o) + RD_2(\phi_3 - \phi_o)]$$

$$+\frac{1}{24}\,\frac{B}{LR}(R - L)[LD_4(\phi_1 - \phi_o) + RD_3(\phi_3 - \phi_o)]$$

$$+\frac{1}{96}\,\frac{T}{LR}(R+L)[LD_1(\phi^1 - \phi_1 - \phi_2 + \phi_o) - RD_2(\phi^2 - \phi_3 - \phi_2 + \phi_o)]$$

$$+\frac{1}{96}\,\frac{B}{LR}(R+L)[LD_4(\phi^4 - \phi_1 - \phi_4 + \phi_o) - RD_3(\phi^3 - \phi_3 - \phi_4 + \phi_o)]$$

$$+\frac{1}{24}\,\frac{r_iR}{TB}[BD_1(\phi^1 - \phi_1 - \phi_2 + \phi_o) + TD_4(\phi^4 - \phi_1 - \phi_4 + \phi_o)]$$

$$+\frac{1}{24}\,\frac{r_iL}{TB}[BD_2(\phi^2 - \phi_3 - \phi_2 + \phi_o) + TD_3(\phi^3 - \phi_3 - \phi_4 + \phi_o)]$$

$$+\frac{1}{24}\,\frac{r_iT}{LR}[LD_1(\phi^1 - \phi_1 - \phi_2 + \phi_o) + RD_2(\phi^2 - \phi_3 - \phi_2 + \phi_o)]$$

$$+\frac{1}{24}\,\frac{r_iB}{LR}[RD_3(\phi^3 - \phi_3 - \phi_4 + \phi_o) + LD_4(\phi^4 - \phi_1 - \phi_4 + \phi_o)]$$

$$+ \frac{1}{4} \Sigma_1 \; T \; P \; \phi_o$$

$$+ \frac{1}{4} \Sigma_2 \; T \; P' \; \phi_o$$

$$+ \frac{1}{4} \Sigma_3 \; B \; P' \; \phi_o$$

$$+ \frac{1}{4} \Sigma_4 \; B \; P \; \phi_o$$

$$- \frac{1}{16} \Sigma_1 \; r_i \; TR(\phi_1 + \phi_2 - 2\phi_o)$$

$$- \frac{1}{16} \Sigma_2 \; r_i \; TL(\phi_3 + \phi_2 - 2\phi_o)$$

$$- \frac{1}{16} \Sigma_3 \; r_i \; BL(\phi_3 + \phi_4 - 2\phi_o)$$

$$- \frac{1}{16} \Sigma_4 \; r_i \; BR(\phi_1 + \phi_4 - 2\phi_o)$$

$$+ O(h^4) = 0 \qquad (36)$$

The finite difference Eq. 36 may be written in the form

$$a_{i,j}\phi_o + b_{i,j}\phi_1 + c_{i,j}\phi_2 + d_{i,j}\phi_3 + e_{i,j}\phi_4 + f_{i,j}\phi^1 + g_{i,j}\phi^2 + h_{i,j}\phi^3 + k_{i,j}\phi^4 + O(h^4) = 0$$

or

$$a_{i,j}\phi_{i,j} + b_{i,j}\phi_{i+1,j} + c_{i,j}\phi_{i,j+1} + d_{i,j}\phi_{i-1,j} + e_{i,j}\phi_{i,j-1}$$

$$+ f_{i,j}\phi_{i+1,j+1} + g_{i,j}\phi_{i-1,j+1} + h_{i,j}\phi_{i-1,j-1}$$

$$+ k_{i,j}\phi_{i+1,j-1} + O(h^4) = 0 \qquad (37)$$

The flux coefficients may be written for the equation $-\nabla^2\phi(r,z) + \Sigma(r,z)\phi(r,z) = 0$, where $\Sigma(r,z) = \dfrac{\Sigma_{ac}(r,z)-\nu\Sigma_f(r,z)}{D_c(r,z)}$ for a reactor and $\Sigma(r,z) = \dfrac{\Sigma_{ae}(r,z)}{D_e(r,z)}$ for a reactor reflector, as follows:

$$K_{i,j} = -D_{i,j-1}\left\{2\left[2h_{r_i}^2(3r_i + h_{r_i}) + 3h_{z_{j-1}}^2(2r_i + h_{r_i})\right]\right.$$

$$\left. + h_{z_{j-1}}^2(h_{r_i} + h_{r_{i-1}}) + 4r_i(h_{r_i}^2 + h_{z_{j-1}}^2)\right\}/96h_{r_i}h_{z_{j-1}}$$

$$h_{i,j} = -D_{i-1,j}\left\{2\left[2h_{r_{i-1}}^2(3r_i - h_{r_{i-1}}) + 3h_{z_{j-1}}^2(2r_i - h_{r_{i-1}})\right]\right.$$

$$\left. - h_{z_j}^2(h_{r_i} + h_{r_{i-1}}) + 4r_i(h_{r_{i-1}}^2 + h_{z_{j-1}}^2)\right\}/96h_{r_{i-1}}h_{z_{j-1}}$$

$$g_{i,j} = -D_{i,j+1}\left\{2\left[2h_{r_{i-1}}^2(3r_i - h_{r_{i-1}}) + 3h_{z_j}^2(2r_i - h_{r_{i-1}})\right]\right.$$

$$\left. - h_{z_j}^2(h_{r_i} + h_{r_{i-1}}) + 4r_i(h_{r_{i-1}}^2 + h_{z_j}^2)\right\}/96h_{z_j}h_{r_{i-1}}$$

$$f_{i,j} = -D_{i,j}\left\{2\left[2h_{r_i}^2(3r_i + h_{r_i}) + 3h_{z_j}^2(2r_i + h_{r_i})\right]\right.$$

$$\left. + h_{z_j}^2(h_{r_i} + h_{r_{i-1}}) + 4r_i(h_{r_i}^2 + h_{z_j}^2)\right\}/96h_{r_i}h_{z_j}$$

$$e_{i,j} = -h_{i,j} - k_{i,j} - \left[4r_i(D_{i-1,j}\,h_{r_{i-1}} + D_{i,j-1}\,h_{r_i})\right.$$

$$\left. +(D_{i,j-1}h_{r_i}^2 - D_{i-1,j}h_{r_{i-1}}^2)\right]/8h_{z_{j-1}}$$

$$+ r_i h_{z_{j-1}}(\Sigma_{i-1,j}h_{r_{i-1}} + \Sigma_{i,j-1}\,h_{r_i})/12$$

$$d_{i,j} = -g_{i,j} - h_{i,j} -(12r_i - 7h_{r_{i-1}} + h_{r_i})(D_{i,j+1}h_{z_j}$$

$$+ D_{i-1,j}\,h_{z_{j-1}})/24h_{r_{i-1}} + r_i h_{r_{i-1}}(\Sigma_{i,j+1}\,h_{z_j}$$

$$+ \Sigma_{i-1,j}\,h_{z_{j-1}})/12r_i h_{r_{i-1}}$$

$$c_{i,j} = -f_{i,j} - g_{i,j} - \Big[4r_i(D_{i,j}h_{r_i} + D_{i-1,j}h_{r_{i-1}})$$

$$+ (D_{i,j}h^2_{r_i} - D_{i-1,j}h^2_{r_{i-1}})\Big] /8h_{z_j}$$

$$+ r_i h_{z_j}(\Sigma_{i,j}h_{r_i} + \Sigma_{i-1,j}\, h_{r_{i-1}})/12$$

$$b_{i,j} = - f_{i,j} - k_{i,j} - (12r_i + 7h_{r_i} - h_{r_{i-1}})(D_{i,j}h_{z_j}$$

$$+ D_{i,j-1}h_{z_{j-1}})/24h_{r_i} + r_i h_{r_i}(\Sigma_{i,j}h_{z_j} + \Sigma_{i,j-1}h_{z_{j-1}})/12$$

$$a_{i,j} = - (b_{i,j} + c_{i,j} + d_{i,j} + e_{i,j} + f_{i,j} + h_{i,j} + g_{i,j} + k_{i,j})$$

$$- h_{r_i}(\Sigma_{i,j}h_{z_j} + \Sigma_{i,j-1}h_{z_{j-1}})(2r_i + h_{r_i})/16$$

$$- h_{r_{i-1}}(\Sigma_{i-1,j}h_{z_j} + \Sigma_{i-1,j}\, h_{z_{j-1}})(2r_i - h_{r_{i-1}})/16$$

## F. The Five-Point Formula

The five-point formula in r-z geometry can be written as [8]

$$b_{i,j}\phi_{i+1,j} + c_{i,j}\phi_{i,j+1} + d_{i,j}\phi_{i-1,j} + e_{i,j}\phi_{i,j-1} + a_{i,j}\phi_{i,j}=0 \qquad (38)$$

where

$$b_{i,j} = -(2r_i + h_{r_i})(D_{i,j-1}h_{z_{j-1}} + D_{i,j}h_{z_j})/4h_{r_i}$$

$$c_{i,j} = -(D_{i,j}h_{r_i} + D_{i-1,j}h_{r_{i-1}})[4r_i + (D_{i,j}h_{r_i}h_{r_{i-1}})]/8h_{z_j}$$

$$d_{i,j} = -(2r_i - h_{r_{i-1}})(D_{i-1,j}h_{z_{j-1}} + D_{i,j+1}h_{z_j})/4h_{r_{i-1}}$$

$$e_{i,j} = -(D_{i,j-1}h_{r_i} + D_{i-1,j}h_{r_{i-1}})[4r_i + (D_{i,j-1}h_{r_i} - D_{i-1,j}h_{r_{i-1}})]/8h_{z_{j-1}}$$

$$a_{i,j} = -(b_{i,j} + c_{i,j} + d_{i,j} + e_{i,j}) + [h_{r_i}(4r_i + h_{r_i})(\Sigma_{i,j}h_{z_j} + \Sigma_{i,j-1}h_{z_{j-1}}) + h_{r_{i-1}}(4r_i - h_{r_{i-1}})(\Sigma_{i,j+1}h_{z_j} + \Sigma_{i-1,j}h_{z_{j-1}})]/16$$

## G. The Analytical Solution

The analytical solution of the neutron diffusion equation over regions with material interfaces is difficult to be obtained even in the simplest cases [5]. To assure the corrections and accuracy of the numerical solution of the finite difference equations, it is useful to solve the neutron diffusion equation analytically over the region R for certain model problems. This technique gives a suitable comparison between the numerical and analytical solution of the neutron diffusion equation. With this comparison, the error range between the two solutions can be determined.

The distribution of neutrons in the core and reflector of a one energy group system may be suitably described by the following equations [11]

$$\nabla^2\phi_c(r,z) + B_c^2(r,z) = 0 \tag{39}$$

$$\nabla^2\phi_e(r,z) - \frac{1}{L_e^2}\,\phi_e(r,z) = 0 \tag{40}$$

where

$$B_c^2 = \frac{\nu\,\Sigma_f - \Sigma_a^{(c)}}{D_c}$$

$$L_e^2 = \frac{D_e}{\Sigma_a^{(e)}}$$

$e \equiv$ Reflector

$C \equiv$ Core

The analytical solutions of Eqs. 39 and 40 for radial and axial reflectors in cylindrical geometry (Figs. 2 and 3) have been determined [11] and may be written as follows:

a: radial-reflector (Fig. 2)

$$\phi_c(r,z) = \phi_{co}J_o(\alpha_c r)\cos\frac{\pi}{H}z \tag{41}$$

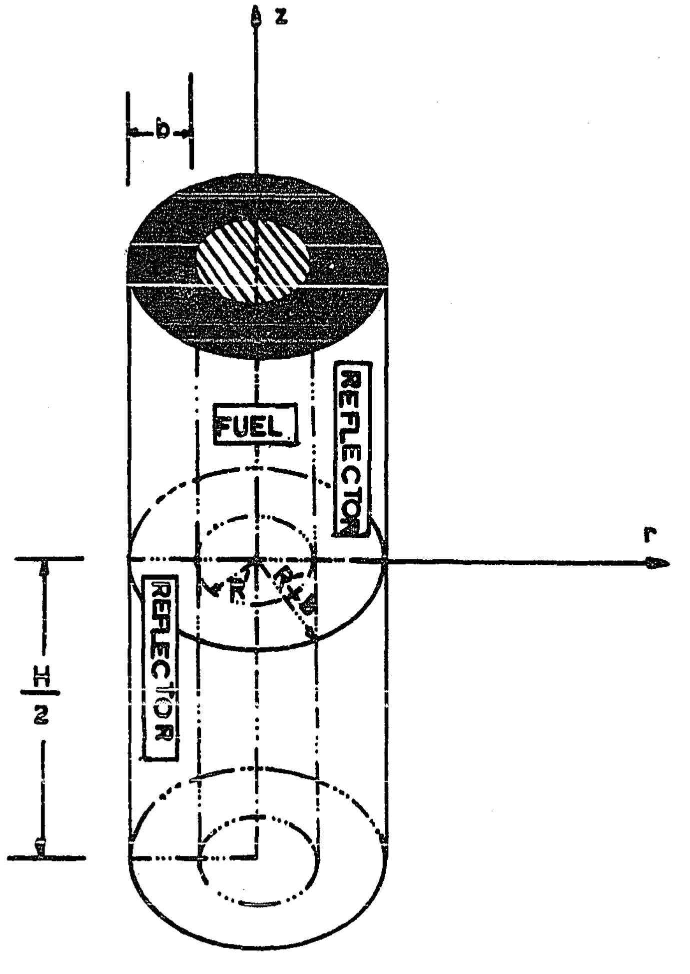


Fig. 2. Radially-reflected cylindrical reactor core

$$\phi_e(r,z) = \phi_{co} J_o(\alpha_c R') \frac{I_o[\alpha_e(R'+b)]K_o(\alpha_e r) - K_o[\alpha_e(R'+b)]I_o(\alpha_e r)}{I_o[\alpha_e(R'+b)]K_o(\alpha_e R') - K_o[\alpha_e(R'+b)]I_o(\alpha_e R')} \cos\frac{\pi}{H}z \tag{42}$$

where $\alpha_c$ is to be determined from the following equation:

$$\alpha_c D_c \frac{J_1(\alpha_c R')}{J_o(\alpha_c R')} = \alpha_e D_e \frac{I_o[\alpha_e(R'+b)]K_1(\alpha_e R') + K_o[\alpha_e(R'+b)]I_1(\alpha_e R')}{I_o[\alpha_e(R'+b)]K_o(\alpha_e R') - K_o[\alpha_e(R'+b)]I_o(\alpha_e R')} \tag{43}$$

$\alpha_e$ and $B_c$ can be calculated from the following relations:

$$\alpha_e^2 = \frac{\Sigma_a^{(e)}}{D_e} + \left(\frac{\pi}{H}\right)^2 \tag{44}$$

$$B_c^2 = \alpha_c^2 + \left(\frac{\pi}{H}\right)^2 \tag{45}$$

b: Axial-reflector (Fig. 3)

$$\phi_c(r,z) = \phi_{co} J_o(\alpha r) \cos \beta_c z \tag{46}$$

$$\phi_e(r,z) = \phi_{co} \frac{\cos(\beta_c H/2)}{\sinh(\beta_e b)} J_o(\alpha r) \sinh[\beta_e(\frac{H}{2} - z)] \tag{47}$$

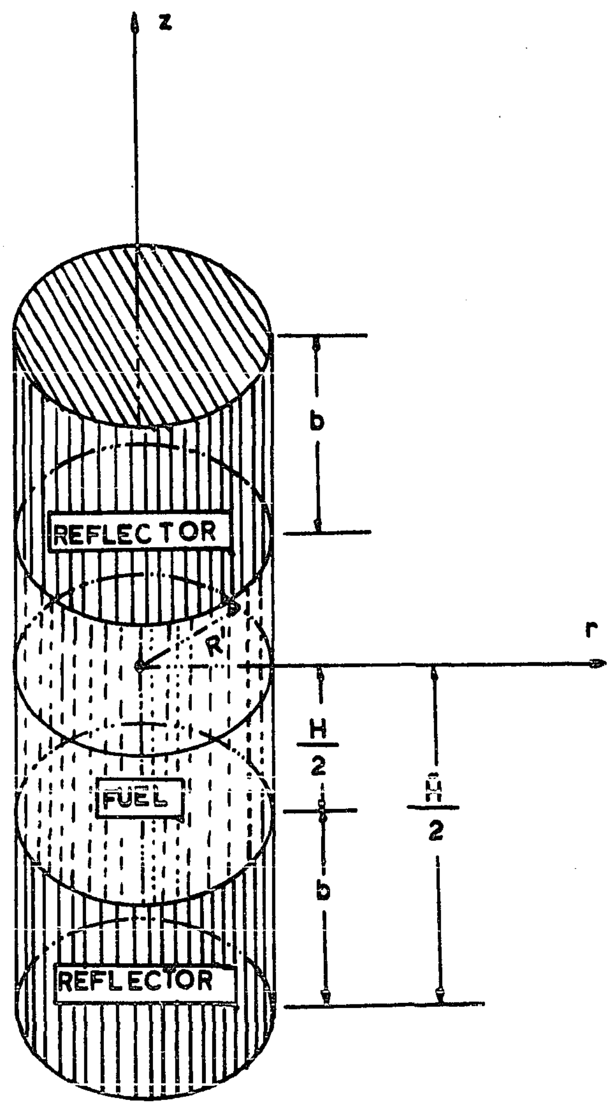


Fig. 3. Axially-reflected cylindrical reactor

where $\beta_c$ is to be determined from the solution of the following equation:

$$\frac{H}{2}\beta_c \tan(\beta_c H/2) = \frac{D_e}{D_c}\frac{H}{2}\beta_e \coth[\beta_e(\frac{\tilde{H}}{2} - \frac{H}{2})] \quad (48)$$

$\alpha$ and $\beta_e$ can be calculated from the following relations:

$$\alpha^2 = \frac{2.405}{R'} \quad (49)$$

$$\beta_e^2 = \frac{\Sigma_a^{(e)}}{D_e} + \alpha^2 \quad (50)$$

where

$\tilde{H} = \frac{H}{2} + b$

$R' \equiv$ Radius of the cylinder

$H \equiv$ The height of the cylinder

$b \equiv$ reflector thickness

# III. RESULTS AND DISCUSSION

The results of this work were obtained by solving Eqs. 37 and 38 numerically and Eqs. 39 and 40 analytically using a computer program. Within the computer program of the analytical solutions Eqs. 43 and 48 are solved numerically using Newton's iterative method [12].

The computational technique is concerned with iterative methods in solving the nine-point and five-point finite difference equations [13,14]. The iterative method, specifically the overrelaxation method, is used because it is easily programmed and has a comparatively fast convergence rate.

Various cylindrical reactor core types such as bare cores, radially reflected cores, and axially reflected cores were considered. The calculated results for the previous configurations cover equal (Figs. 4, 5, and 6) and nonequal (Figs. 7, 8, and 9) spacing along radial and axial axes.

The neutron flux values for equal spacing along the r- and z- axes for the nine-point, five-point, and analytical solution are tabulated in Tables 1, 2, and 3. The absolute error[1] and

---

[1]Absolute error = numerical solution-analytical solution.

Euclidean norm[2] for the radial and axial cases are computed as listed in Tables 4, 5, 6, 7, 8, and 9. Graphs of absolute error versus radial and axial distances are plotted as shown in Figs. 10-32. Plots of residual vector norm[3] versus iteration are shown in Figs. 33-35. From the plots of Figs. 10-32 one can observe the accuracy of the nine-point formulation over the five-point formulation.

The over all Euclidean norm for the nine-point and five-point solutions are 0.0053 and 0.1103 (bare cylindrical core), 0.0109 and 0.1433 (radial-reflected cylindrical core), and 0.0075 and 0.01720 (axial-reflected cylindrical core), respectively. The comparisons between the Euclidean norms for

---

[2]The Euclidean norm is [2]

$$\ell_2 = \left[ \sum_i |E_i|^2 \right]^{\frac{1}{2}} ,$$

where $E_i$ is the absolute error at a mesh point, (i = 1,2,3,4-----).

[3]The residual vector may be defined as [9]

$$r^{(P)} = x^{(P+1)} - x^{(P)}$$

where r = the residual vector
x = the solution
P = iteration number, (P = 0,1,2,3,----).

Thus, the residual vector norm may be defined as [2]

$$\left[ \sum_i (r_i)^2 \right]^{\frac{1}{2}}$$

the nine-point and five-point results illustrate the accuracy of the nine-point formulation over the five-point formulation.

The convergence rates[4] as calculated from the slopes of the curves Figs. 33-35 for the nine-point and five-point formulas are 0.654 and 0.448 (bare cylindrical core), 0.464 and 0.304 (radially reflected cylindrical core), and 0.406 and 0.412 (axially reflected cylindrical core), respectively. These results show that the nine-point formula converges faster than the five-point formula for these cases. The saw-tooth effect in the curves of Figs. 34 and 35 represents the inner iteration over the eigenvalue $(\Sigma(r,z)/D(r,z))$ of the boundary value equation until criticality was achieved.

---

[4] The convergence rate may be defined as [9]

$$\nu = - \ln \lambda_1$$

where

$\nu$ = the convergence rate

$\lambda_1$ = the largest eigenvalue of the iteration matrix.

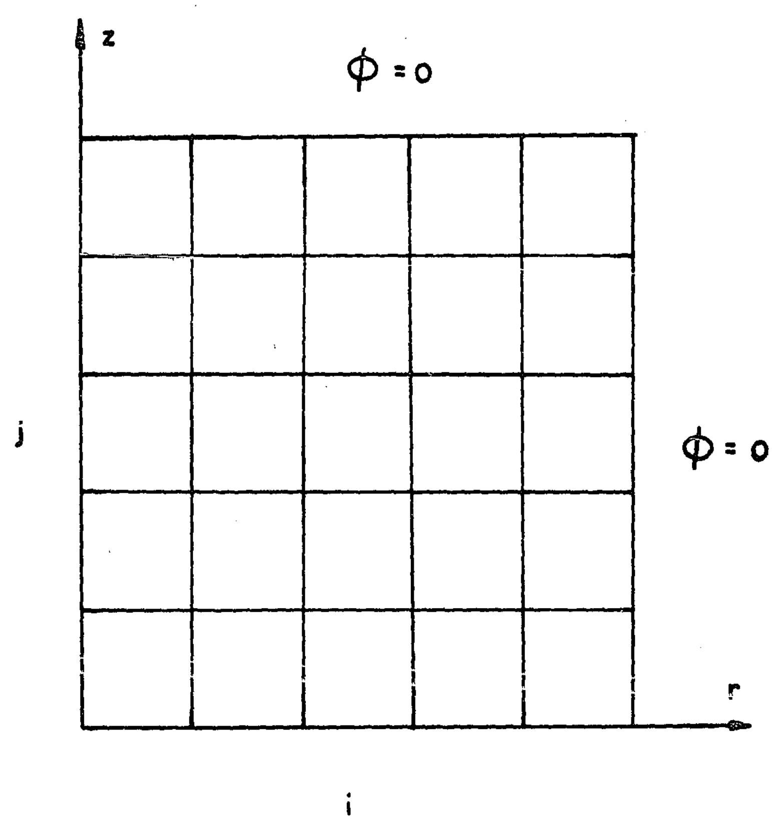


Fig. 4. Equal spacing mesh for a cylindrical bare reactor core

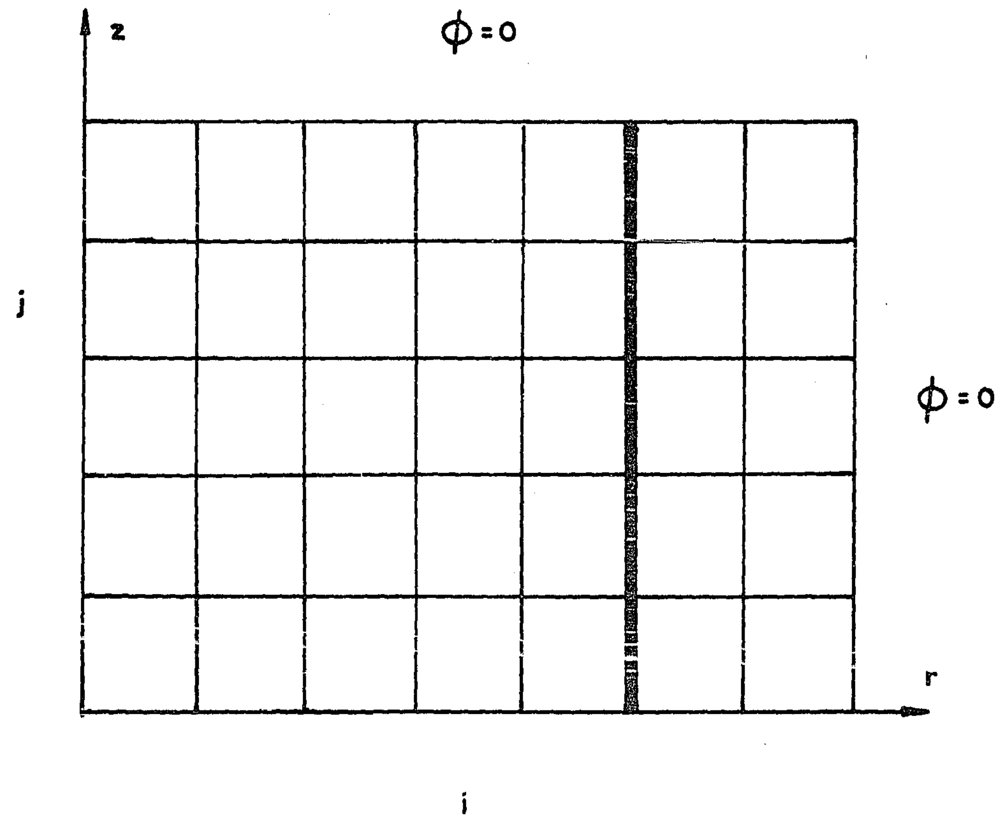


Fig. 5. Equal spacing mesh for a radially reflected reactor core

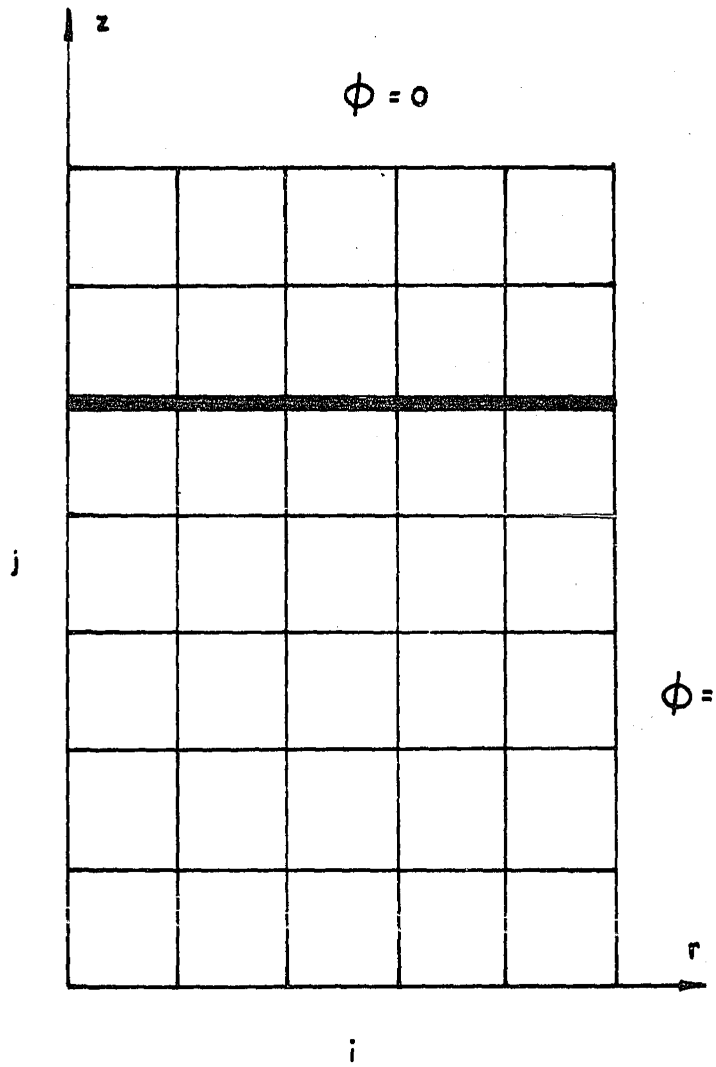


Fig. 6. Equal spacing mesh for axially reflected cylindrical reactor core

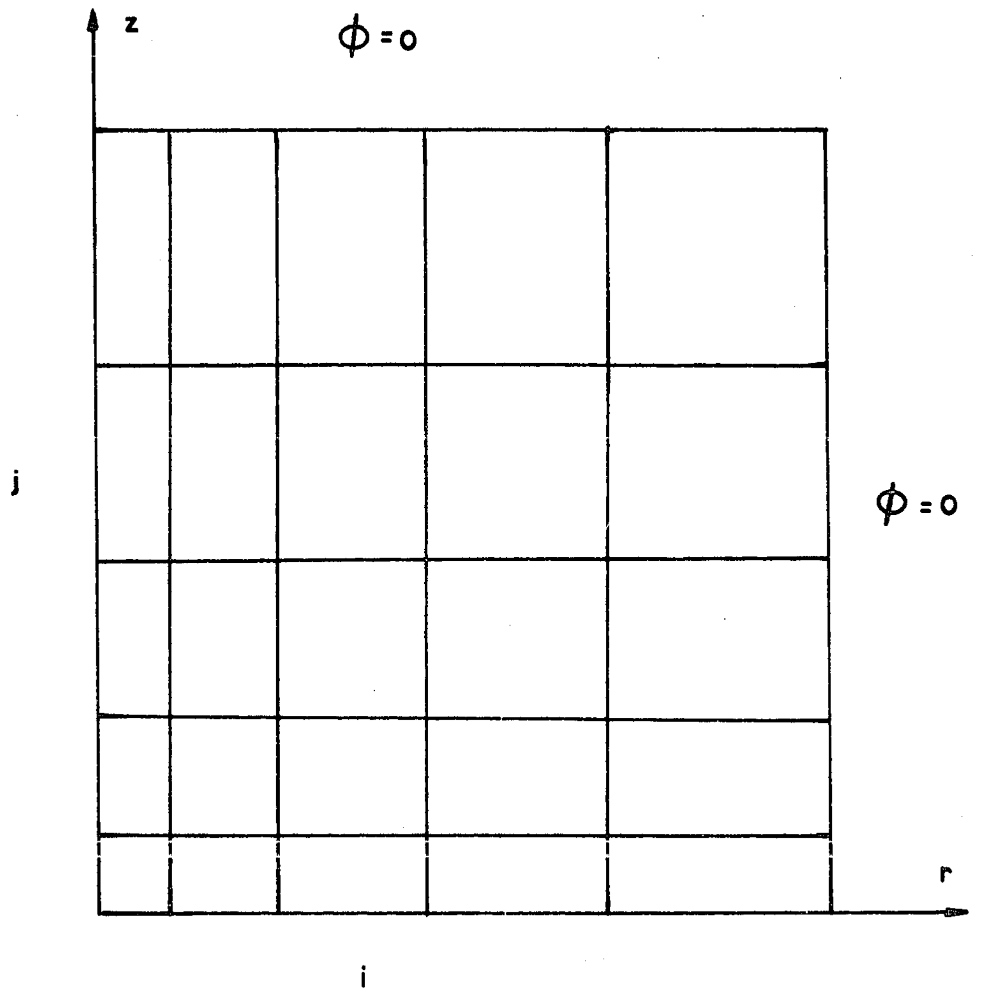


Fig. 7. Unequal spacing mesh for a cylindrical bare reactor core

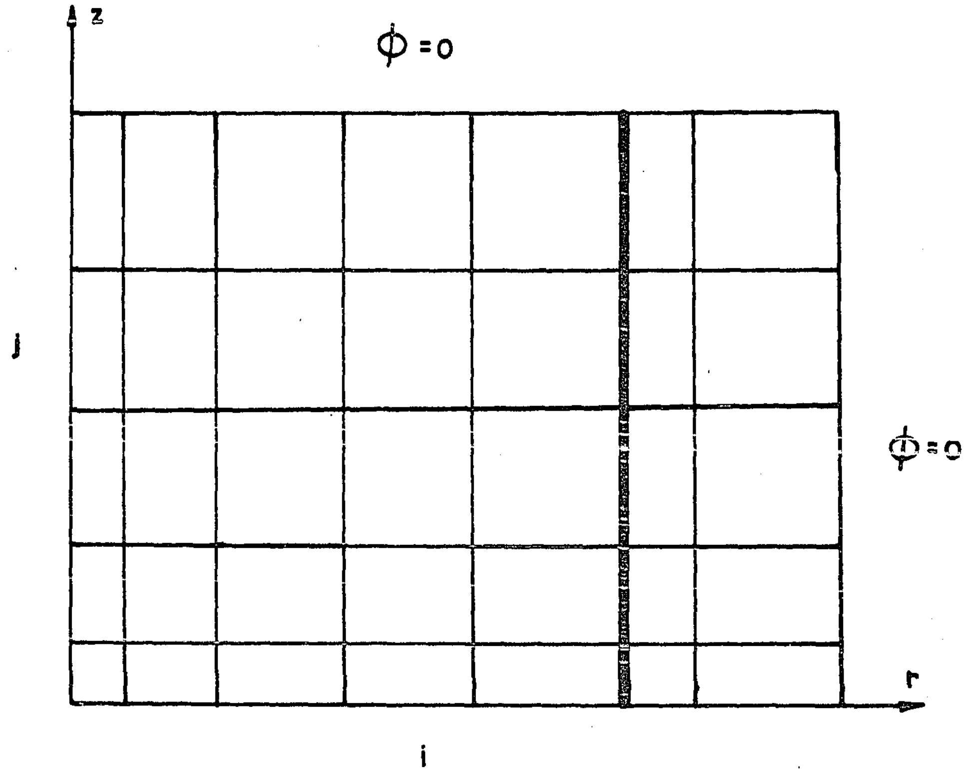


Fig. 8. Unequal spacing for a radially reflected reactor core

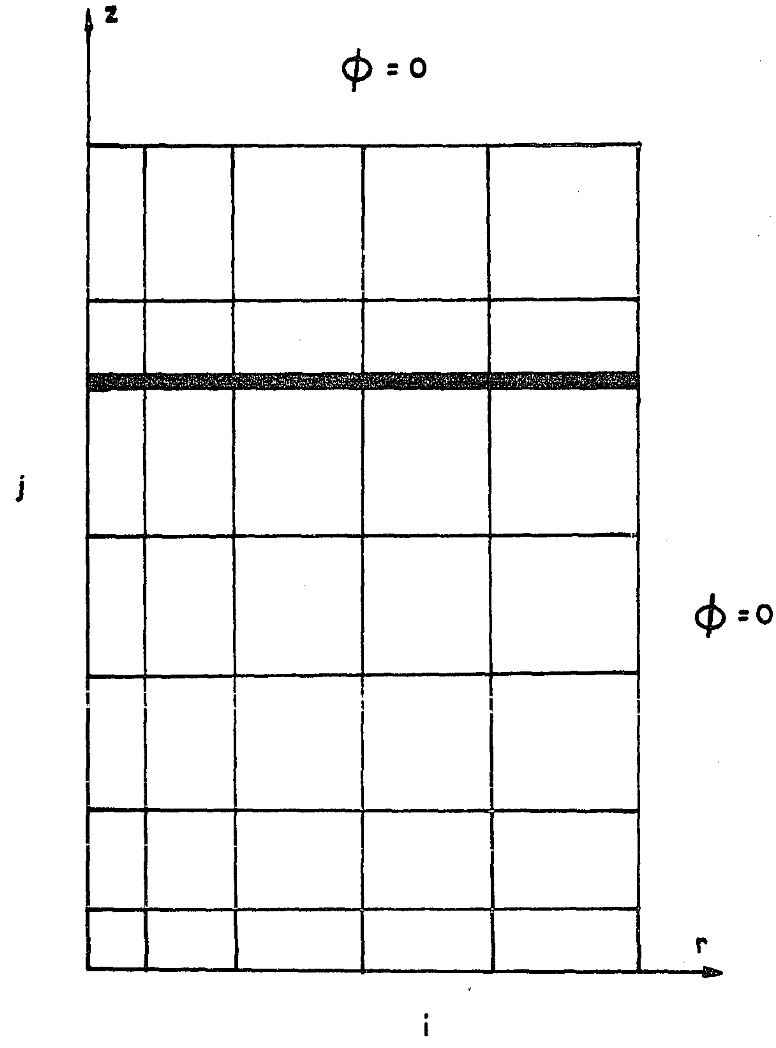


Fig. 9. Unequal spacing mesh for axially reflected cylindrical reactor core

Table 1. The neutron flux distribution in a bare reactor core for equal spacing along the radial and axial axes and for input data $D_c$ = 1.0 cm, $\Sigma_c$ = 0.33 $cm^{-1}$, and R' = H/2 = 5.0 cm

| Method | z \ r | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| ANS[a] | 0 | 1.000 | 0.943 | 0.782 | 0.544 | 0.268 | 0.000 |
| | 1 | 0.951 | 0.897 | 0.744 | 0.517 | 0.255 | 0.000 |
| | 2 | 0.809 | 0.763 | 0.632 | 0.440 | 0.217 | 0.000 |
| | 3 | 0.588 | 0.554 | 0.460 | 0.320 | 0.158 | 0.000 |
| | 4 | 0.309 | 0.291 | 0.242 | 0.168 | 0.083 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NPS[b] | 0 | 1.000 | 0.942 | 0.783 | 0.545 | 0.270 | 0.000 |
| | 1 | 0.951 | 0.896 | 0.744 | 0.519 | 0.257 | 0.000 |
| | 2 | 0.809 | 0.763 | 0.633 | 0.442 | 0.219 | 0.000 |
| | 3 | 0.588 | 0.554 | 0.460 | 0.321 | 0.159 | 0.000 |
| | 4 | 0.309 | 0.292 | 0.242 | 0.168 | 0.084 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS[c] | 0 | 1.000 | 0.895 | 0.731 | 0.506 | 0.249 | 0.000 |
| | 1 | 0.955 | 0.854 | 0.698 | 0.483 | 0.238 | 0.000 |
| | 2 | 0.815 | 0.730 | 0.596 | 0.412 | 0.203 | 0.000 |
| | 3 | 0.595 | 0.532 | 0.435 | 0.301 | 0.148 | 0.000 |
| | 4 | 0.314 | 0.281 | 0.230 | 0.159 | 0.078 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

[a]ANS = analytical solution.

[b]NPS = nine-point solution.

[c]FPS = five-point solution.

Table 2. The flux distribution in radially reflected core for equal spacing and for input data of $D_c = 0.5$ cm, $\Sigma_c = 0.1385$ cm$^{-1}$, R = z = 5 cm, $D_{R_c} = 1.0$ cm, $\Sigma_{R_c} = 0.1$ cm$^{-1}$, and b = 2 cm

| Method | z \ r | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| ANS | 0 | 1.000 | 0.956 | 0.830 | 0.637 | 0.405 | 0.160 | 0.067 | 0.000 |
| | 1 | 0.951 | 0.909 | 0.789 | 0.606 | 0.385 | 0.153 | 0.064 | 0.000 |
| | 2 | 0.809 | 0.773 | 0.671 | 0.516 | 0.327 | 0.130 | 0.054 | 0.000 |
| | 3 | 0.588 | 0.562 | 0.488 | 0.375 | 0.238 | 0.094 | 0.039 | 0.000 |
| | 4 | 0.309 | 0.295 | 0.456 | 0.197 | 0.125 | 0.050 | 0.021 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NPS | 0 | 1.000 | 0.955 | 0.830 | 0.639 | 0.408 | 0.165 | 0.068 | 0.000 |
| | 1 | 0.951 | 0.908 | 0.789 | 0.608 | 0.388 | 0.157 | 0.065 | 0.000 |
| | 2 | 0.809 | 0.773 | 0.671 | 0.518 | 0.330 | 0.133 | 0.055 | 0.000 |
| | 3 | 0.588 | 0.561 | 0.488 | 0.376 | 0.240 | 0.097 | 0.040 | 0.000 |
| | 4 | 0.309 | 0.295 | 0.256 | 0.198 | 0.126 | 0.051 | 0.021 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.915 | 0.782 | 0.595 | 0.376 | 0.146 | 0.061 | 0.000 |
| | 1 | 0.951 | 0.870 | 0.744 | 0.566 | 0.356 | 0.139 | 0.058 | 0.000 |
| | 2 | 0.809 | 0.740 | 0.633 | 0.482 | 0.303 | 0.118 | 0.049 | 0.000 |
| | 3 | 0.588 | 0.538 | 0.460 | 0.350 | 0.220 | 0.086 | 0.036 | 0.000 |
| | 4 | 0.309 | 0.283 | 0.242 | 0.184 | 0.116 | 0.045 | 0.019 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 3. The flux distribution in axially reflected core as for equal spacing and input data $D_c = 0.5$ cm $\Sigma_c = 0.1536$ cm$^{-1}$, R = z = 5 cm, $D_{R_c} = 1.0$ cm, $\Sigma_{R_c} = 0.1$, and b = 2 cm

| Method | r \ z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| ANS | 0 | 1.000 | 0.962 | 0.852 | 0.678 | 0.452 | 0.192 | 0.082 | 0.000 |
| | 1 | 0.943 | 0.907 | 0.803 | 0.639 | 0.426 | 0.181 | 0.077 | 0.000 |
| | 2 | 0.782 | 0.752 | 0.666 | 0.530 | 0.353 | 0.150 | 0.064 | 0.000 |
| | 3 | 0.543 | 0.523 | 0.463 | 0.368 | 0.246 | 0.104 | 0.045 | 0.000 |
| | 4 | 0.268 | 0.258 | 0.228 | 0.182 | 0.121 | 0.052 | 0.022 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NPS | 0 | 1.000 | 0.962 | 0.852 | 0.677 | 0.451 | 0.188 | 0.081 | 0.000 |
| | 1 | 0.942 | 0.406 | 0.802 | 0.637 | 0.424 | 0.180 | 0.076 | 0.000 |
| | 2 | 0.781 | 0.751 | 0.665 | 0.528 | 0.352 | 0.149 | 0.063 | 0.000 |
| | 3 | 0.544 | 0.523 | 0.463 | 0.368 | 0.245 | 0.104 | 0.044 | 0.000 |
| | 4 | 0.269 | 0.259 | 0.229 | 0.182 | 0.121 | 0.051 | 0.022 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.962 | 0.851 | 0.674 | 0.447 | 0.185 | 0.080 | 0.000 |
| | 1 | 0.891 | 0.857 | 0.758 | 0.601 | 0.398 | 0.165 | 0.071 | 0.000 |
| | 2 | 0.726 | 0.698 | 0.617 | 0.489 | 0.324 | 0.134 | 0.058 | 0.000 |
| | 3 | 0.500 | 0.481 | 0.425 | 0.337 | 0.223 | 0.092 | 0.040 | 0.000 |
| | 4 | 0.246 | 0.236 | 0.209 | 0.166 | 0.110 | 0.045 | 0.020 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 4. The calculated results of absolute errors and Euclidean norms for equal spacing in a bare core at different axial levels

| z cm | r cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.003 | 0.0053 | 0.0000 | 0.082 | 0.1103 |
| | 1 | 0.8356 | | | 48.3252 | | |
| | 2 | 0.5273 | | | 50.6976 | | |
| | 3 | 1.8940 | | | 37.6814 | | |
| | 4 | 1.9957 | | | 18.7239 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1 | 0 | 0.1614 | 0.003 | | 3.7467 | 0.012 | |
| | 1 | 0.4543 | | | 42.6084 | | |
| | 2 | 0.7814 | | | 45.2221 | | |
| | 3 | 1.9979 | | | 33.9430 | | |
| | 4 | 1.9949 | | | 16.8740 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 2 | 0 | 0.4057 | 0.003 | | 6.3860 | 0.058 | |
| | 1 | 0.1193 | | | 33.3836 | | |
| | 2 | 0.9005 | | | 36.1296 | | |
| | 3 | 1.8649 | | | 27.2561 | | |
| | 4 | 1.7792 | | | 13.5566 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3 | 0 | 0.4868 | 0.002 | | 6.9725 | 0.097 | |
| | 1 | 0.1120 | | | 22.1675 | | |
| | 2 | 0.8245 | | | 24.5436 | | |
| | 3 | 1.4754 | | | 18.6226 | | |
| | 4 | 1.3523 | | | 9.2679 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 4 | 0 | 0.3376 | 0.0013 | | 4.8964 | 0.0196 | |
| | 1 | 0.1650 | | | 10.5528 | | |
| | 2 | 0.6459 | | | 12.0030 | | |
| | 3 | 0.8389 | | | 9.1678 | | |
| | 4 | 0.7424 | | | 4.5655 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 5. The calculated results of absolute errors and Euclidean norms for equal spacing in a bare core at different radial levels

| r cm | z cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.0007 | 0.0053 | 0.0000 | 0.0113 | 0.1103 |
| | 1 | 0.1614 | | | 3.7467 | | |
| | 2 | 0.4057 | | | 6.3860 | | |
| | 3 | 0.4868 | | | 6.9725 | | |
| | 4 | 0.3376 | | | 4.8964 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1 | 0 | 0.8356 | 0.00098 | | 48.3252 | 0.07660 | |
| | 1 | 0.4543 | | | 42.6089 | | |
| | 2 | 0.1193 | | | 33.3836 | | |
| | 3 | 0.1120 | | | 22.1675 | | |
| | 4 | 0.1650 | | | 10.5528 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 2 | 0 | 0.5273 | 0.00167 | | 50.6976 | 0.08165 | |
| | 1 | 0.7814 | | | 45.2221 | | |
| | 2 | 0.9005 | | | 36.1296 | | |
| | 3 | 0.8245 | | | 24.5436 | | |
| | 4 | 0.6459 | | | 12.003 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3 | 0 | 1.8940 | 0.0037 | | 37.6814 | 0.0612 | |
| | 1 | 1.9979 | | | 33.9430 | | |
| | 2 | 1.8649 | | | 27.2561 | | |
| | 3 | 1.4754 | | | 18.6226 | | |
| | 4 | 0.8389 | | | 9.1678 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 4 | 0 | 1.9957 | 0.0037 | | 18.7239 | 0.0304 | |
| | 1 | 1.9949 | | | 16.8740 | | |
| | 2 | 1.7792 | | | 13.5566 | | |
| | 3 | 1.3523 | | | 9.2679 | | |
| | 4 | 0.7424 | | | 4.5655 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 6. The calculated results of absolute errors and Euclidean norms for equal spacing in a radially reflected core at different axial levels

| z cm | r cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.0063 | 0.0104 | 0.0000 | 0.0830 | 0.1433 |
| | 1 | 1.1444 | | | 41.3011 | | |
| | 2 | 0.0221 | | | 47.6269 | | |
| | 3 | 1.8909 | | | 42.1592 | | |
| | 4 | 3.6502 | | | 29.8813 | | |
| | 5 | 4.3610 | | | 14.4312 | | |
| | 6 | 1.5600 | | | 5.8537 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 1 | 0 | 0.0182 | 0.006 | | 0.1877 | 0.079 | |
| | 1 | 1.0802 | | | 39.1359 | | |
| | 2 | 0.0237 | | | 45.1908 | | |
| | 3 | 1.7912 | | | 40.025 | | |
| | 4 | 3.4646 | | | 28.3779 | | |
| | 5 | 4.1445 | | | 13.7596 | | |
| | 6 | 1.48251 | | | 5.5611 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 2 | 0 | 0.0481 | 0.005 | | 0.3488 | 0.067 | |
| | 1 | 0.9033 | | | 33.1447 | | |
| | 2 | 0.0160 | | | 38.3341 | | |
| | 3 | 1.5213 | | | 33.9750 | | |
| | 4 | 2.9433 | | | 24.0990 | | |
| | 5 | 3.5234 | | | 11.6473 | | |
| | 6 | 1.2600 | | | 4.7248 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 3 | 0 | 0.0589 | 0.004 | | 0.3735 | 0.048 | |
| | 1 | 0.6458 | | | 23.9867 | | |
| | 2 | 0.0092 | | | 27.7811 | | |

Table 6 (Continued)

| z cm | r cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| | 3 | 1.1033 | | | 24.6372 | | |
| | 4 | 2.1357 | | | 17.4816 | | |
| | 5 | 2.5584 | | | 8.4523 | | |
| | 6 | 0.9146 | | | 3.4291 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 4 | 0 | 0.0393 | 0.002 | | 0.2401 | 0.025 | |
| | 1 | 0.3373 | | | 12.5752 | | |
| | 2 | 0.0058 | | | 14.5782 | | |
| | 3 | 0.5777 | | | 12.9335 | | |
| | 4 | 1.1208 | | | 9.1794 | | |
| | 5 | 1.34413 | | | 4.4395 | | |
| | 6 | 0.48045 | | | 1.8010 | | |
| | 7 | 0.0000 | | | 0.0000 | | |

Table 7. The calculated results for the absolute errors and Euclidean norms for equal spacing in a radially reflected core at different radial levels

| r cm | z cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.0001 | 0.0109 | 0.0000 | 0.0006 | 0.1433 |
| | 1 | 0.0182 | | | 0.1877 | | |
| | 2 | 0.0481 | | | 0.3488 | | |
| | 3 | 0.0589 | | | 0.3735 | | |
| | 4 | 0.0393 | | | 0.2401 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1 | 0 | 1.1444 | 0.002 | | 41.3011 | 0.071 | |
| | 1 | 1.0802 | | | 39.1359 | | |
| | 2 | 0.9033 | | | 33.1447 | | |
| | 3 | 0.6458 | | | 23.9867 | | |
| | 4 | 0.3373 | | | 12.5752 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 2 | 0 | 0.0221 | 0.00004 | | 47.6269 | 0.0823 | |
| | 1 | 0.0237 | | | 45.1908 | | |
| | 2 | 0.0160 | | | 38.3341 | | |
| | 3 | 0.0090 | | | 27.7811 | | |
| | 4 | 0.0058 | | | 14.5782 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3 | 0 | 1.8909 | 0.003 | | 42.1592 | 0.073 | |
| | 1 | 1.7912 | | | 40.0250 | | |
| | 2 | 1.5213 | | | 33.9750 | | |
| | 3 | 1.1033 | | | 24.6372 | | |
| | 4 | 0.5777 | | | 12.9335 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 7 (Continued)

| r cm | z cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 4 | 0 | 3.6502 | 0.006 | | 29.8813 | 0.052 | |
| | 1 | 3.4646 | | | 28.3779 | | |
| | 2 | 2.9433 | | | 24.0990 | | |
| | 3 | 2.1357 | | | 17.4816 | | |
| | 4 | 1.1208 | | | 9.1794 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 5 | 0 | 4.3610 | 0.008 | | 14.4312 | 0.025 | |
| | 1 | 4.1445 | | | 13.7596 | | |
| | 2 | 3.5234 | | | 11.6473 | | |
| | 3 | 2.5584 | | | 8.4523 | | |
| | 4 | 1.3441 | | | 4.4395 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 6 | 0 | 1.5600 | 0.003 | | 5.8537 | 0.010 | |
| | 1 | 1.4825 | | | 5.5611 | | |
| | 2 | 1.2600 | | | 4.7248 | | |
| | 3 | 0.9146 | | | 3.4291 | | |
| | 4 | 0.4805 | | | 1.801 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 8. The calculated results for absolute errors and Euclidean norms for equal spacing in a axially reflected core at different radial levels

| r cm | z cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.005 | 0.0075 | 0.0000 | 0.01 | 0.0172 |
| | 1 | 0.0963 | | | 3.3350 | | |
| | 2 | 0.3689 | | | 1.3922 | | |
| | 3 | 0.8128 | | | 3.0825 | | |
| | 4 | 1.2746 | | | 5.1571 | | |
| | 5 | 4.6798 | | | 7.2482 | | |
| | 6 | 1.1765 | | | 2.3044 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 1 | 0 | 1.4785 | 0.004 | | 51.7290 | 0.099 | |
| | 1 | 1.5423 | | | 50.1119 | | |
| | 2 | 1.6462 | | | 45.3807 | | |
| | 3 | 1.7773 | | | 37.8684 | | |
| | 4 | 2.0271 | | | 28.0321 | | |
| | 5 | 1.7173 | | | 16.4291 | | |
| | 6 | 1.3216 | | | 6.3143 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 2 | 0 | 0.7875 | 0.003 | | 55.9785 | 0.106 | |
| | 1 | 0.8650 | | | 54.1634 | | |
| | 2 | 1.0006 | | | 48.8033 | | |
| | 3 | 1.1736 | | | 40.2747 | | |
| | 4 | 1.2854 | | | 29.1294 | | |
| | 5 | 1.2626 | | | 16.0606 | | |
| | 6 | 0.8785 | | | 6.2885 | | |
| | 7 | 0.0000 | | | 0.0000 | | |

Table 8 (Continued)

| r cm | z cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 3 | 0 | 0.4123 | 0.001 | | 43.4105 | 0.082 | |
| | 1 | 0.3193 | | | 41.9898 | | |
| | 2 | 0.1200 | | | 37.7737 | | |
| | 3 | 0.1514 | | | 31.0564 | | |
| | 4 | 0.4468 | | | 22.2801 | | |
| | 5 | 0.7045 | | | 12.0068 | | |
| | 6 | 0.5447 | | | 4.7354 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 4 | 0 | 1.0144 | 0.002 | | 22.4297 | 0.042 | |
| | 1 | 0.9377 | | | 21.6938 | | |
| | 2 | 0.7507 | | | 19.5043 | | |
| | 3 | 0.4761 | | | 16.0128 | | |
| | 4 | 0.1431 | | | 11.4507 | | |
| | 5 | 0.2436 | | | 6.1130 | | |
| | 6 | 0.2317 | | | 2.4182 | | |
| | 7 | 0.0000 | | | 0.0000 | | |

Table 9. The calculated results for absolute errors and Euclidean norms for equal spacing in axially reflected core at different axial levels

| z cm | r cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.002 | 0.0075 | 0.0000 | 0.091 | 0.0172 |
| | 1 | 1.4785 | | | 51.7290 | | |
| | 2 | 0.7875 | | | 55.9785 | | |
| | 3 | 0.4123 | | | 43.4105 | | |
| | 4 | 1.0144 | | | 22.4297 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1 | 0 | 0.0963 | 0.002 | | 3.3350 | 0.088 | |
| | 1 | 1.5423 | | | 50.1119 | | |
| | 2 | 0.8650 | | | 54.1634 | | |
| | 3 | 0.3193 | | | 41.9898 | | |
| | 4 | 0.9377 | | | 21.6938 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 2 | 0 | 0.3689 | 0.002 | | 1.3922 | 0.079 | |
| | 1 | 1.6462 | | | 45.3807 | | |
| | 2 | 1.0006 | | | 48.8033 | | |
| | 3 | 0.1200 | | | 37.7737 | | |
| | 4 | 0.7507 | | | 19.5043 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3 | 0 | 0.8128 | 0.002 | | 3.0825 | 0.066 | |
| | 1 | 1.7773 | | | 37.8684 | | |
| | 2 | 1.1736 | | | 40.2747 | | |
| | 3 | 0.1514 | | | 31.0564 | | |
| | 4 | 0.4761 | | | 16.0128 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 9 (Continued)

| z cm | r cm | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 4 | 0 | 1.2746 | 0.003 | | 5.1571 | 0.048 | |
| | 1 | 2.0271 | | | 28.0321 | | |
| | 2 | 1.2854 | | | 29.1294 | | |
| | 3 | 0.4468 | | | 22.2801 | | |
| | 4 | 0.1431 | | | 11.4507 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 5 | 0 | 4.6798 | 0.005 | | 7.2482 | 0.028 | |
| | 1 | 1.7173 | | | 16.4291 | | |
| | 2 | 1.2626 | | | 16.0606 | | |
| | 3 | 0.7045 | | | 12.0068 | | |
| | 4 | 0.2436 | | | 6.1130 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 6 | 0 | 1.1765 | 0.002 | | 2.3044 | 0.011 | |
| | 1 | 1.3216 | | | 6.3143 | | |
| | 2 | 0.8785 | | | 6.2885 | | |
| | 3 | 0.5447 | | | 4.7354 | | |
| | 4 | 0.2317 | | | 2.4182 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

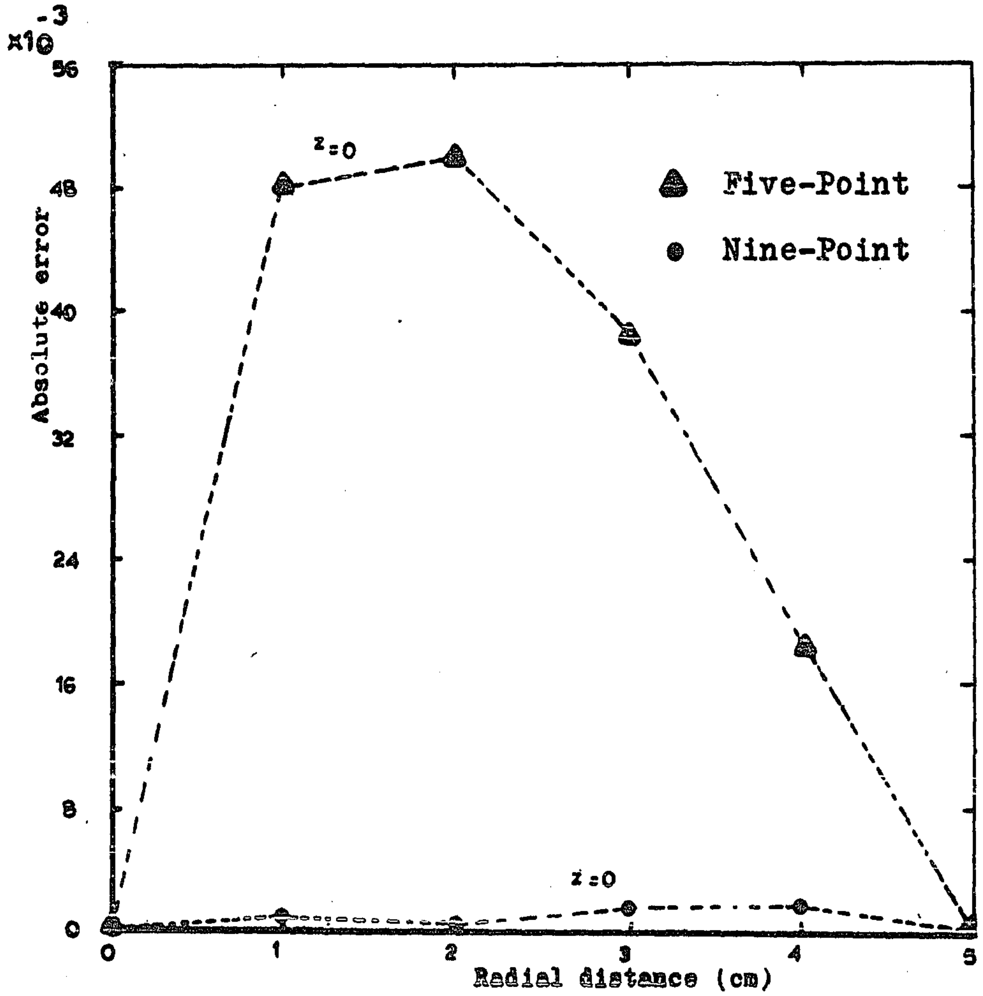


Fig. 10. Absolute error as a function of radial distance for z = 0 and equal spacing in a cylindrical bare reactor core

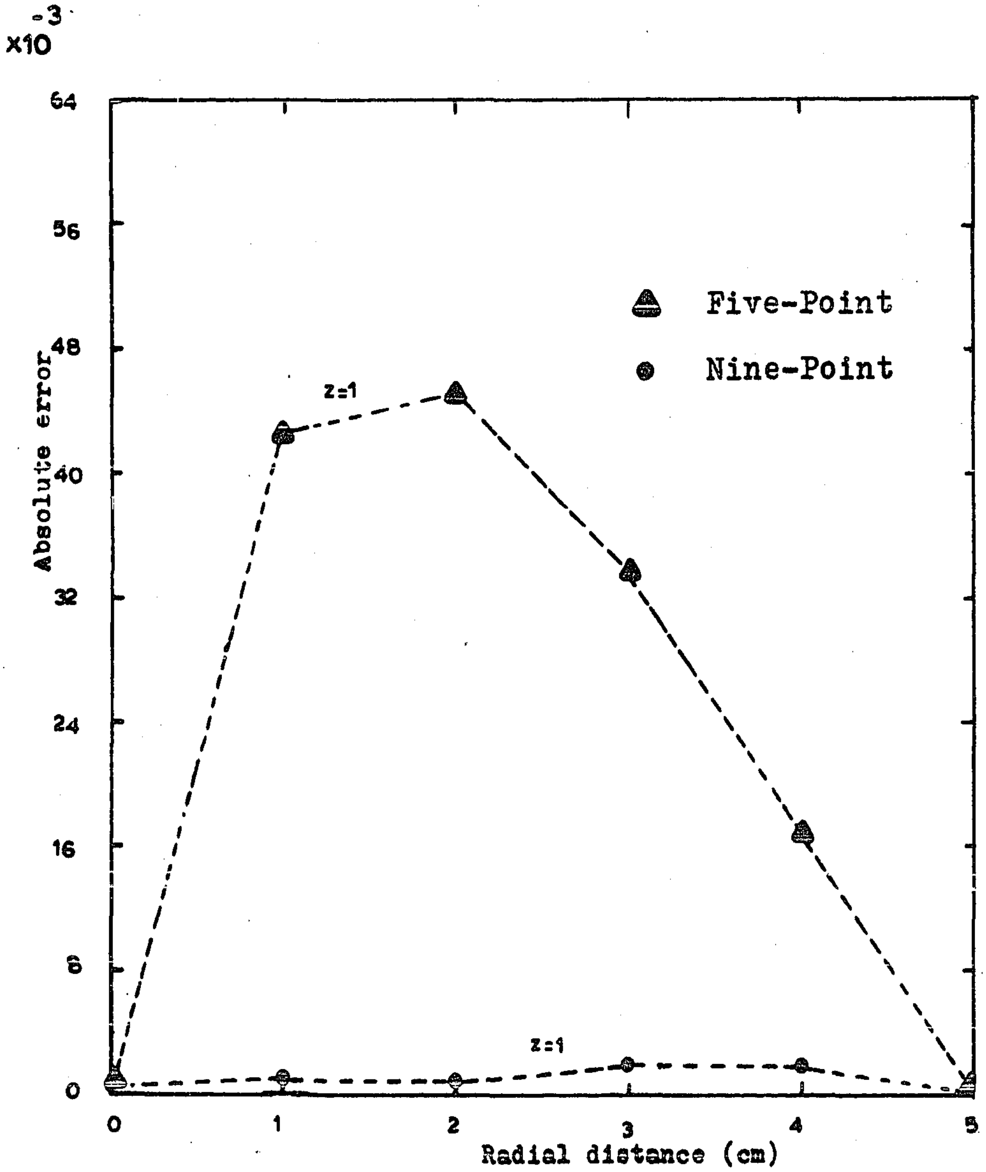


Fig. 11. Absolute error as a function of radial distance for z = 1 and equal spacing in a cylindrical bare reactor core

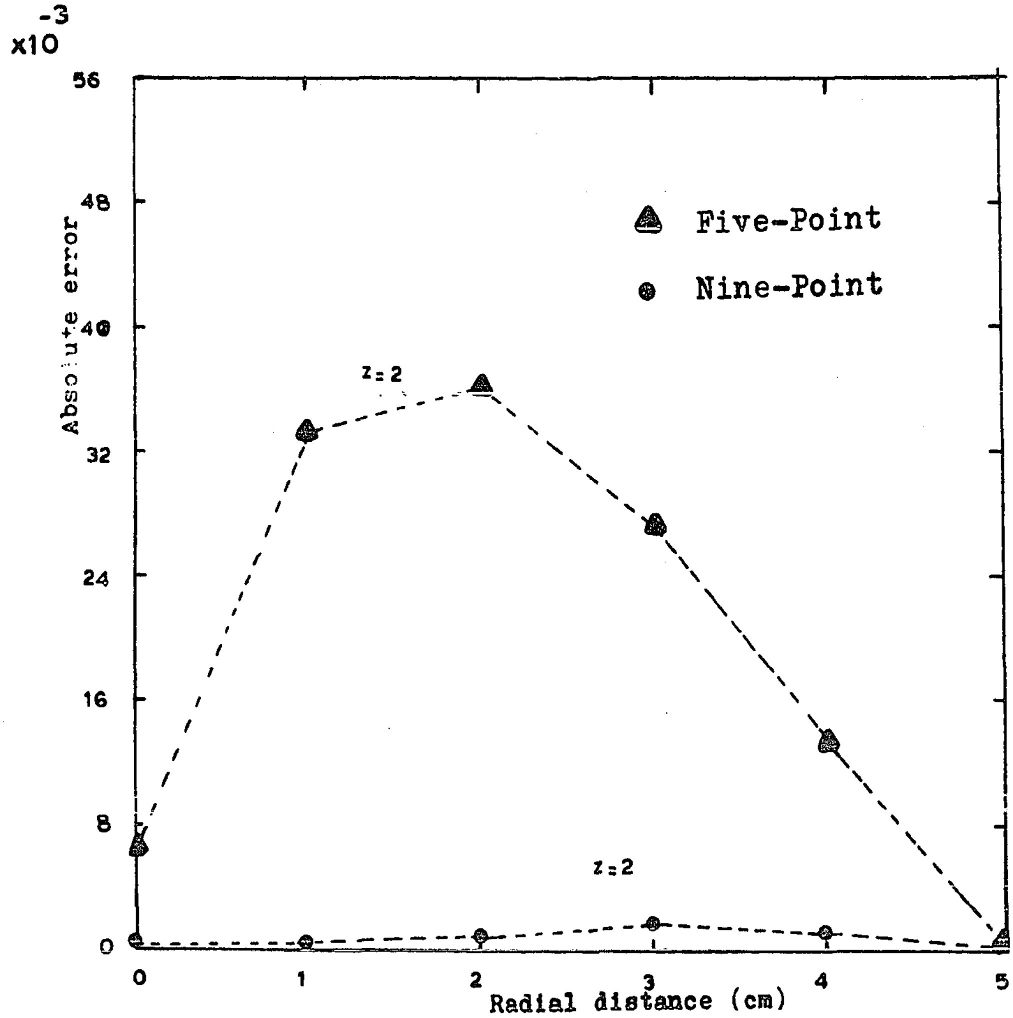


Fig. 12. Absolute error as a function of radial distance for z = 2 and equal spacing in a cylindrical bare reactor core

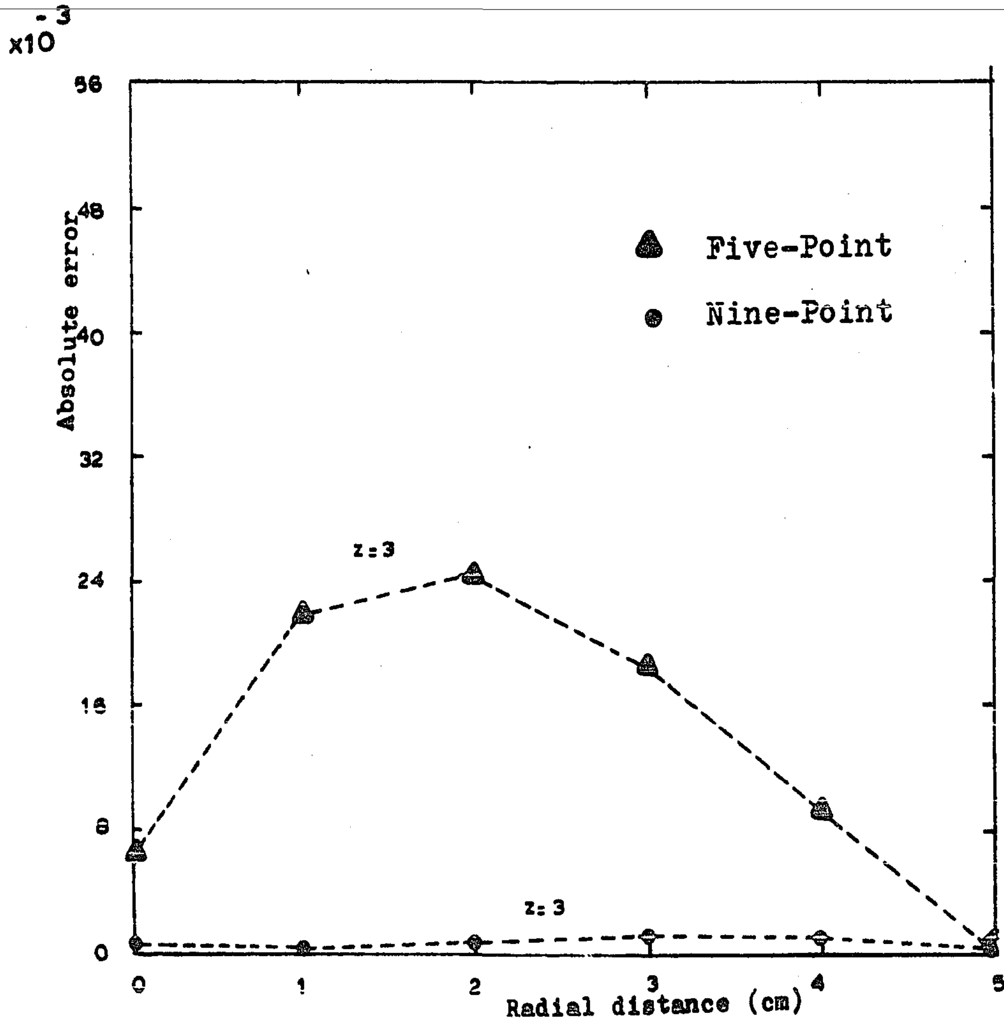


Fig. 13. Absolute error as a function of radial distance for z = 3 and equal spacing in a cylindrical bare reactor core

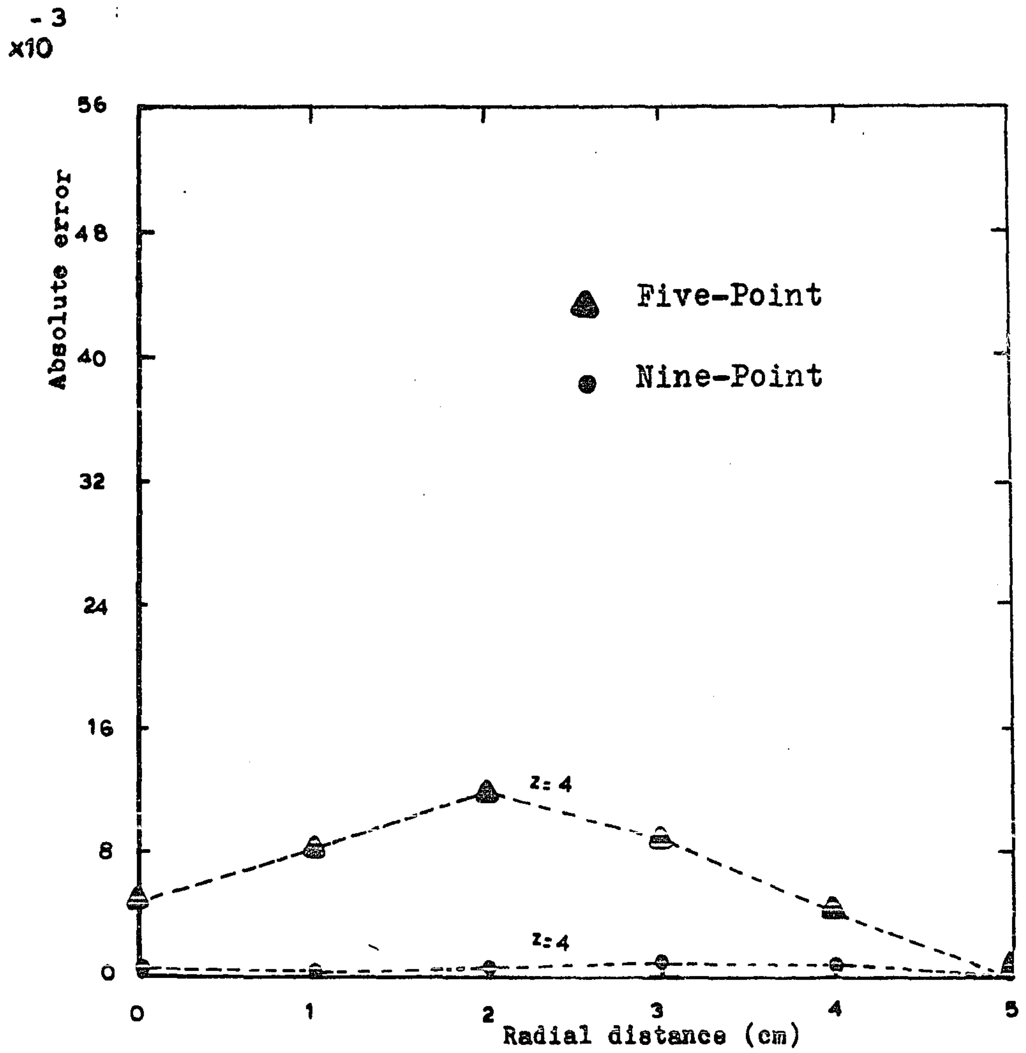


Fig. 14. Absolute error as a function of radial distance for z = 4 and equal spacing in a cylindrical bare reactor core

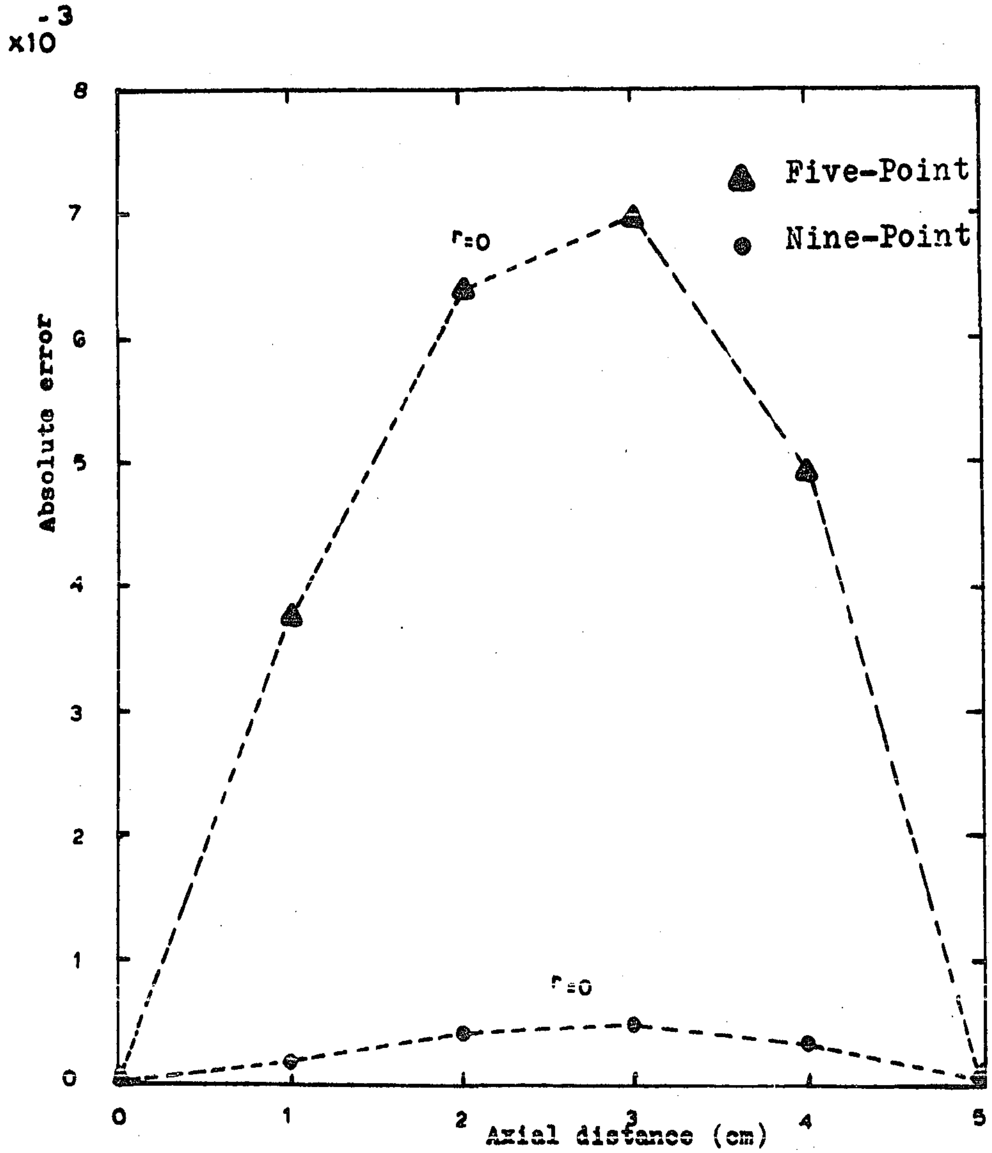


Fig. 15. Absolute error as a function of axial distance for r = 0 and equal spacing in a cylindrical bare reactor core

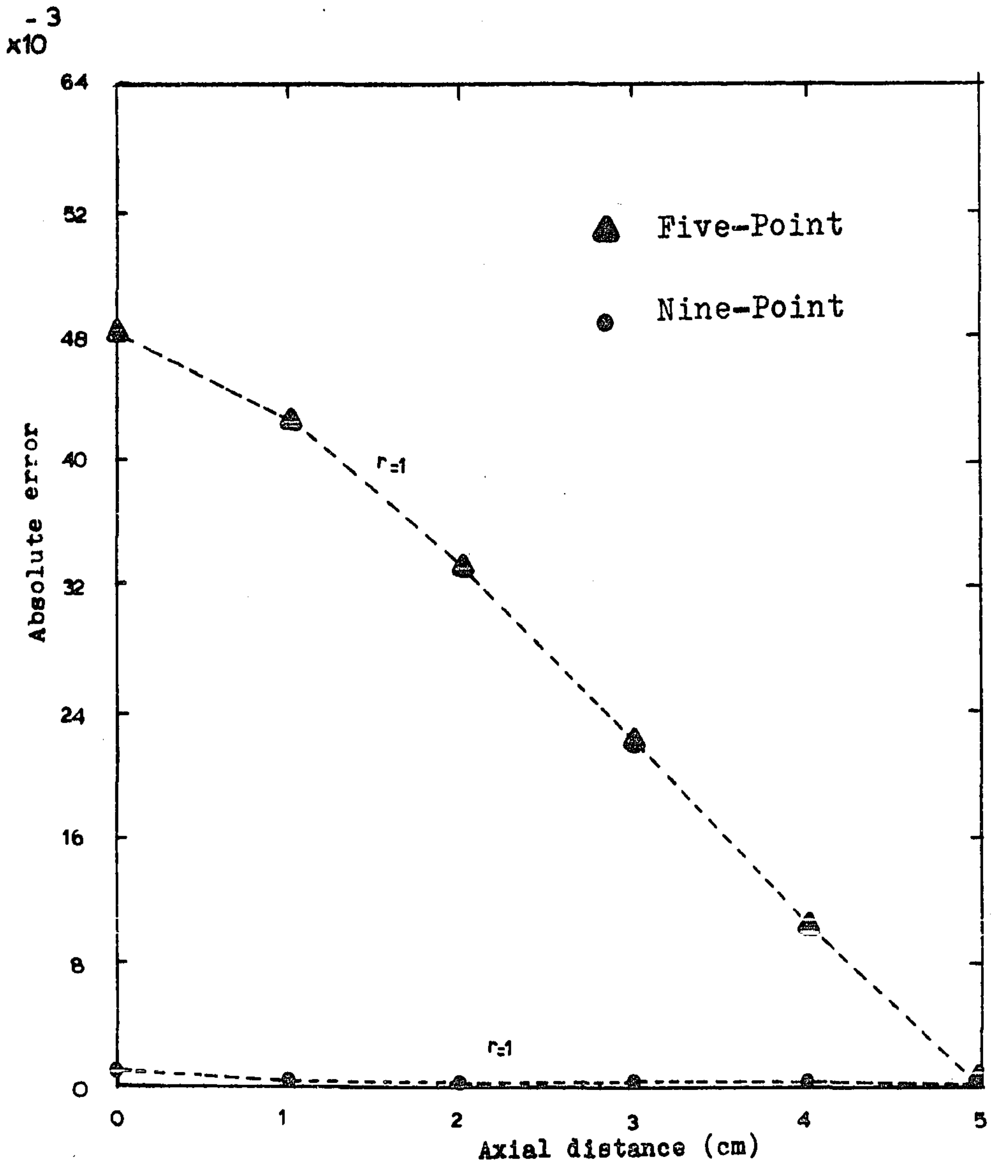


Fig. 16. Absolute error as a function of axial distance for r = 1 and equal spacing in a cylindrical bare reactor core

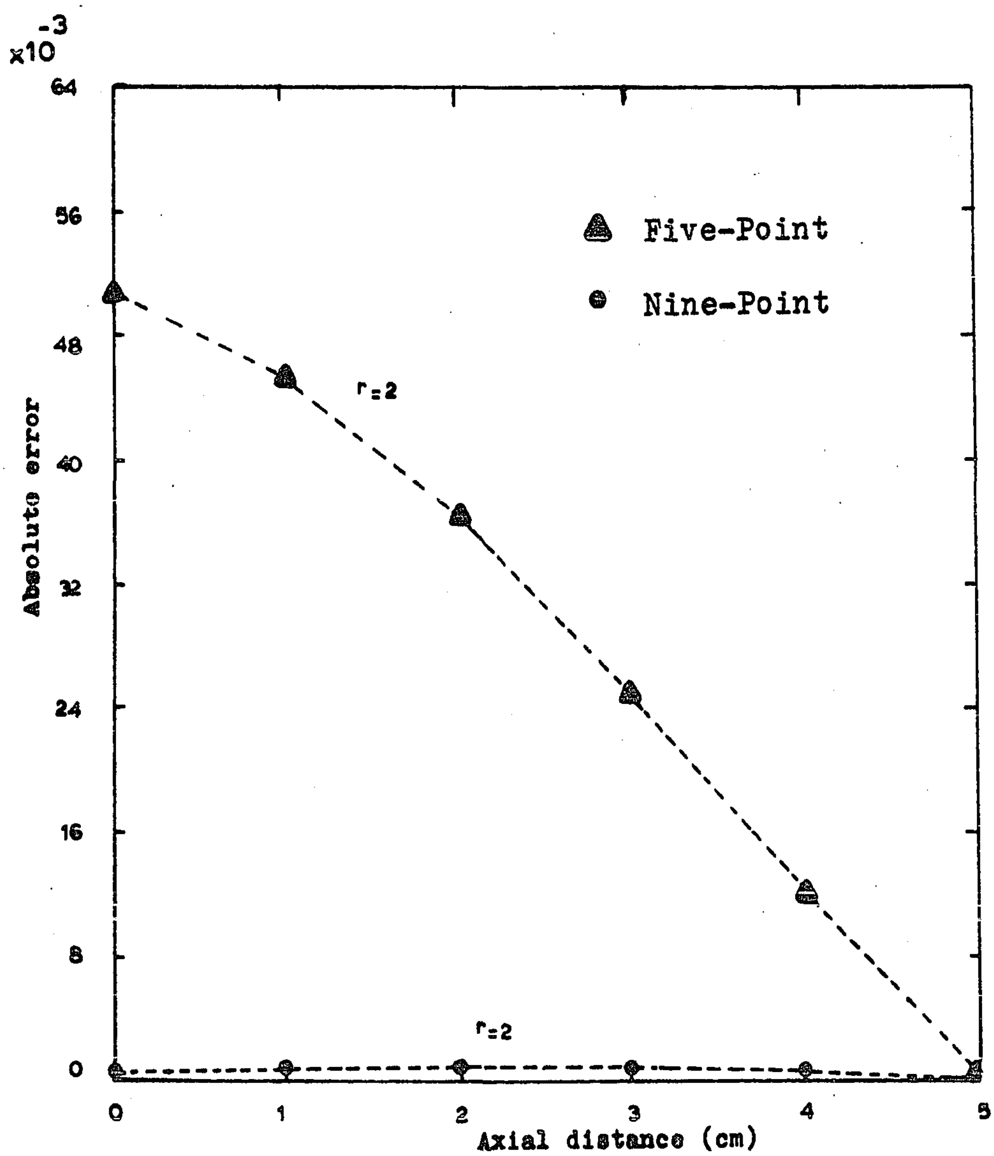


Fig. 17. Absolute error as a function as axial distance for r = 2 and equal spacing in a cylindrical bare reactor core

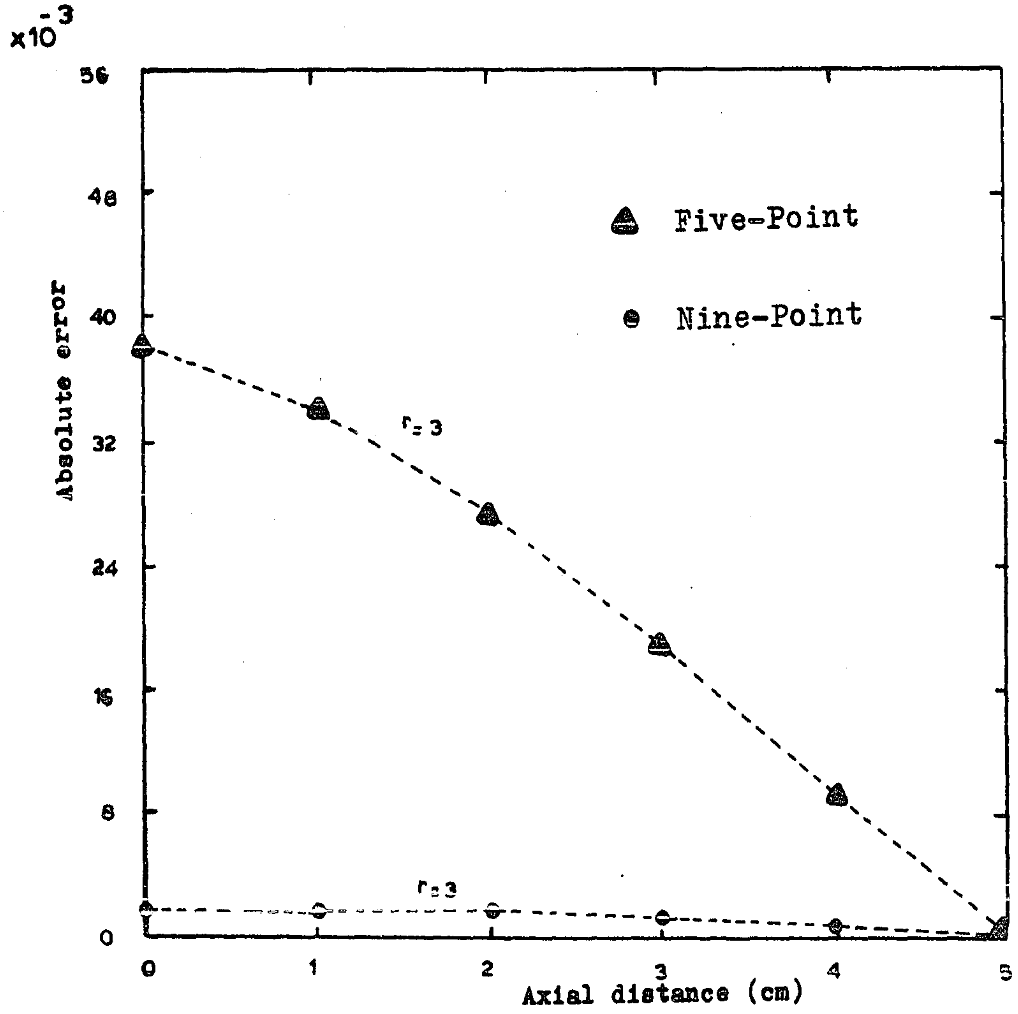


Fig. 18. Absolute error as a function of axial distance for r = 3 and equal spacing in a cylindrical bare reactor core

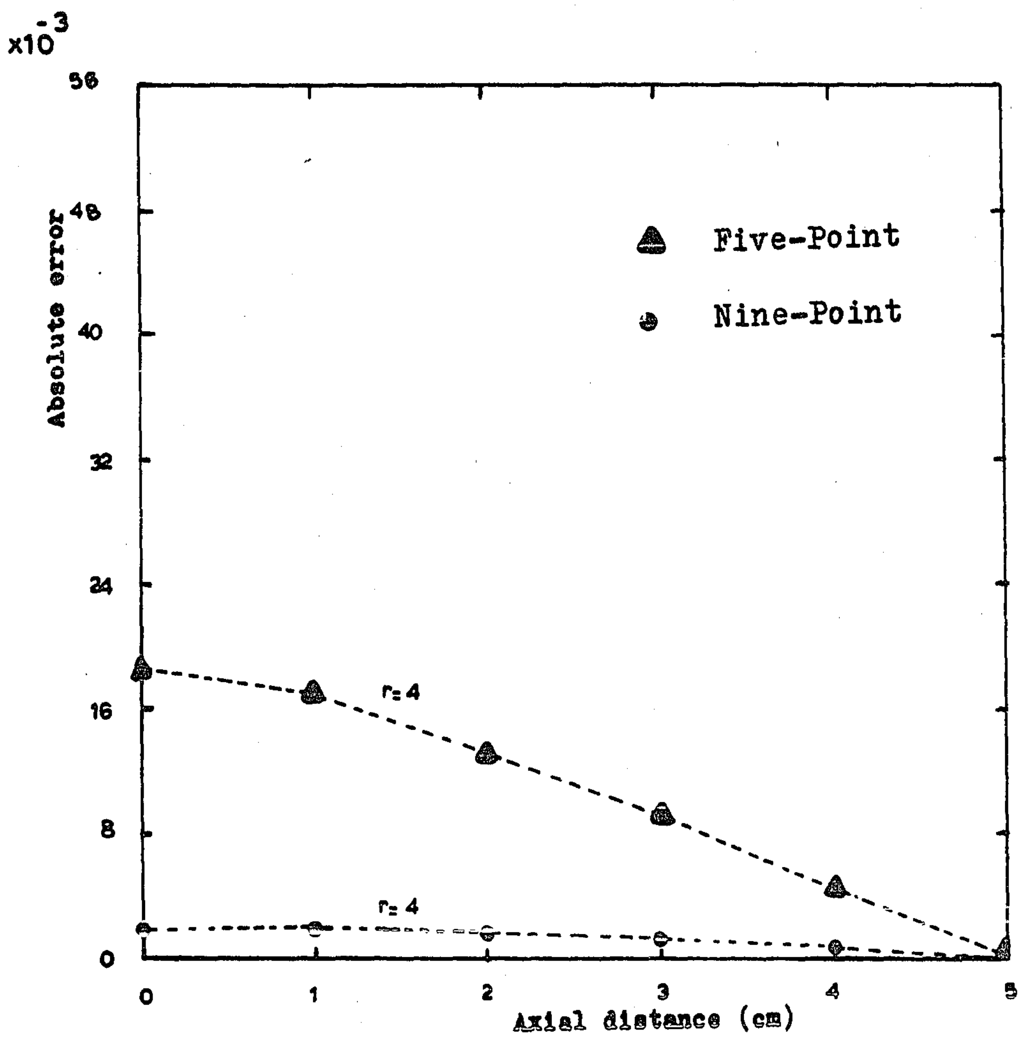


Fig. 19. Absolute error as a function of axial distance for r = 4 and equal spacing in a cylindrical bare reactor core

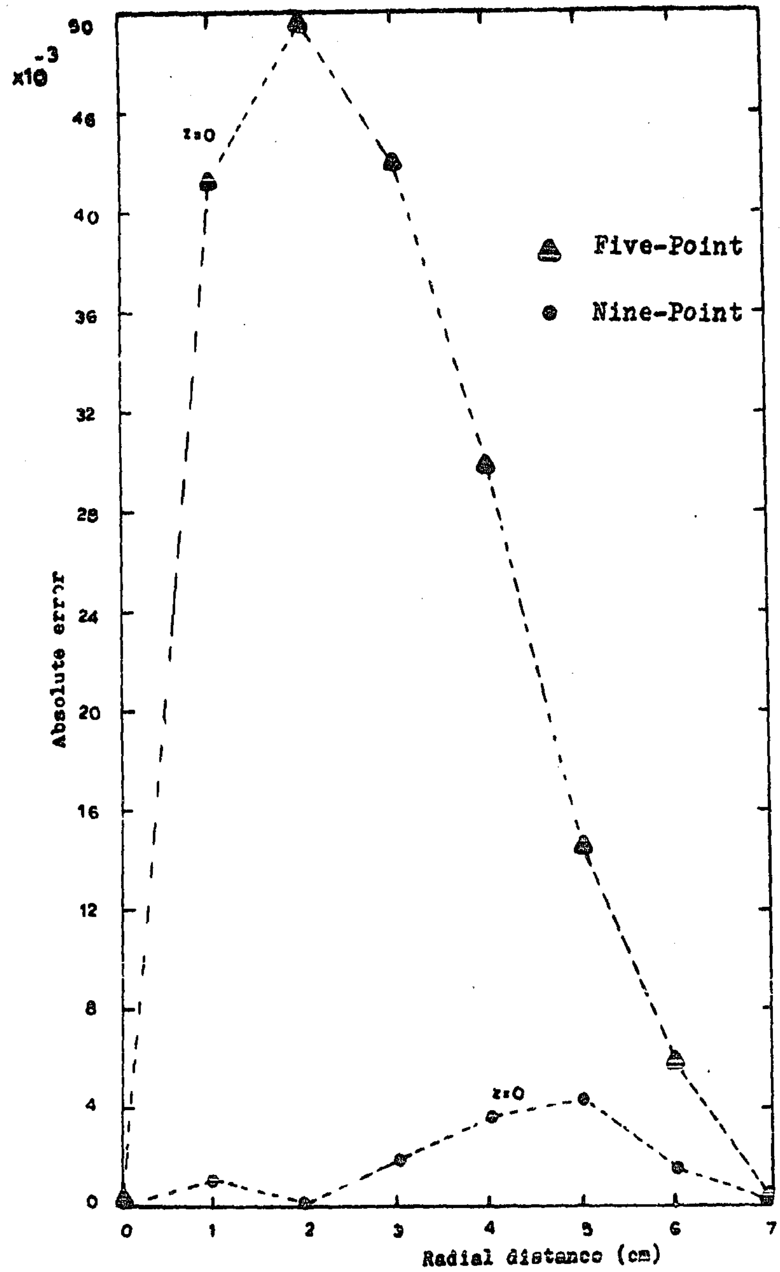


Fig. 20. Absolute error as a function of radial distance for z = 0 and equal spacing in a radially reflected cylindrical reactor core

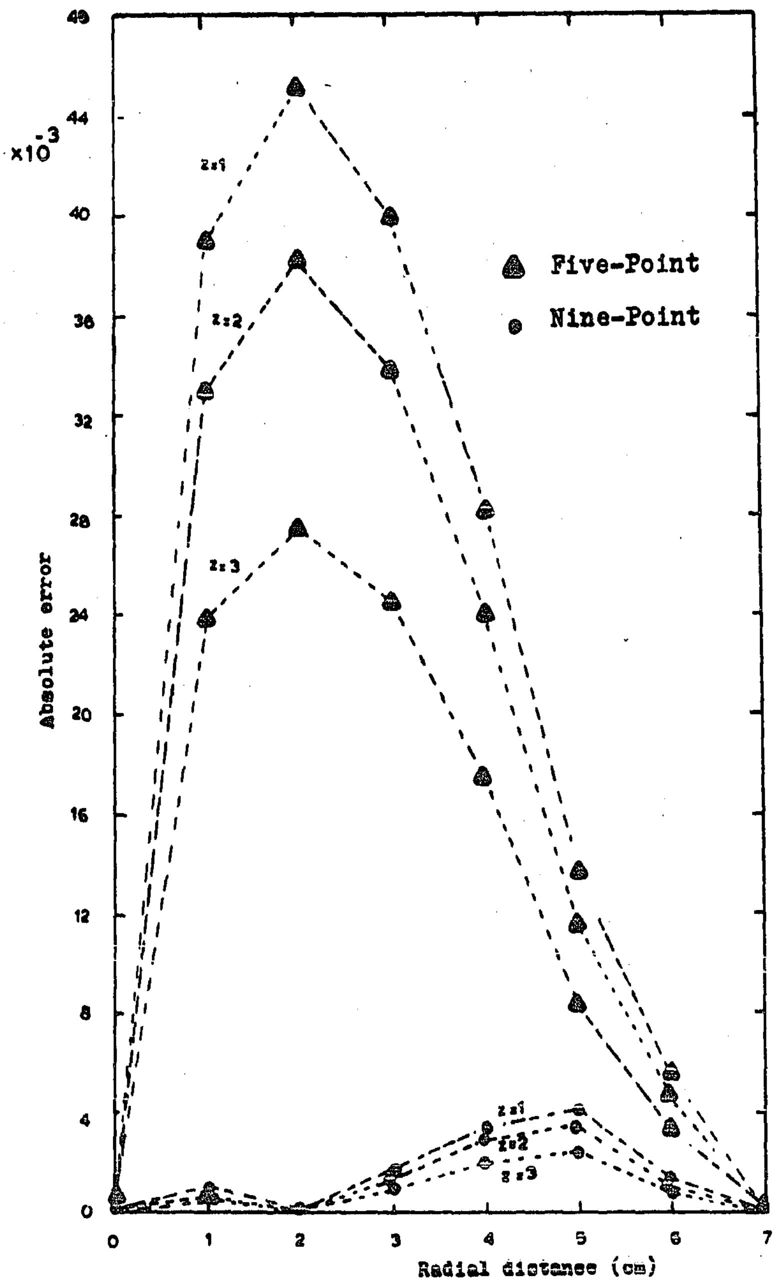


Fig. 21. Absolute error as a function of radial distance for z = 1,2,3, and equal spacing in a radially reflected cylindrical reactor core

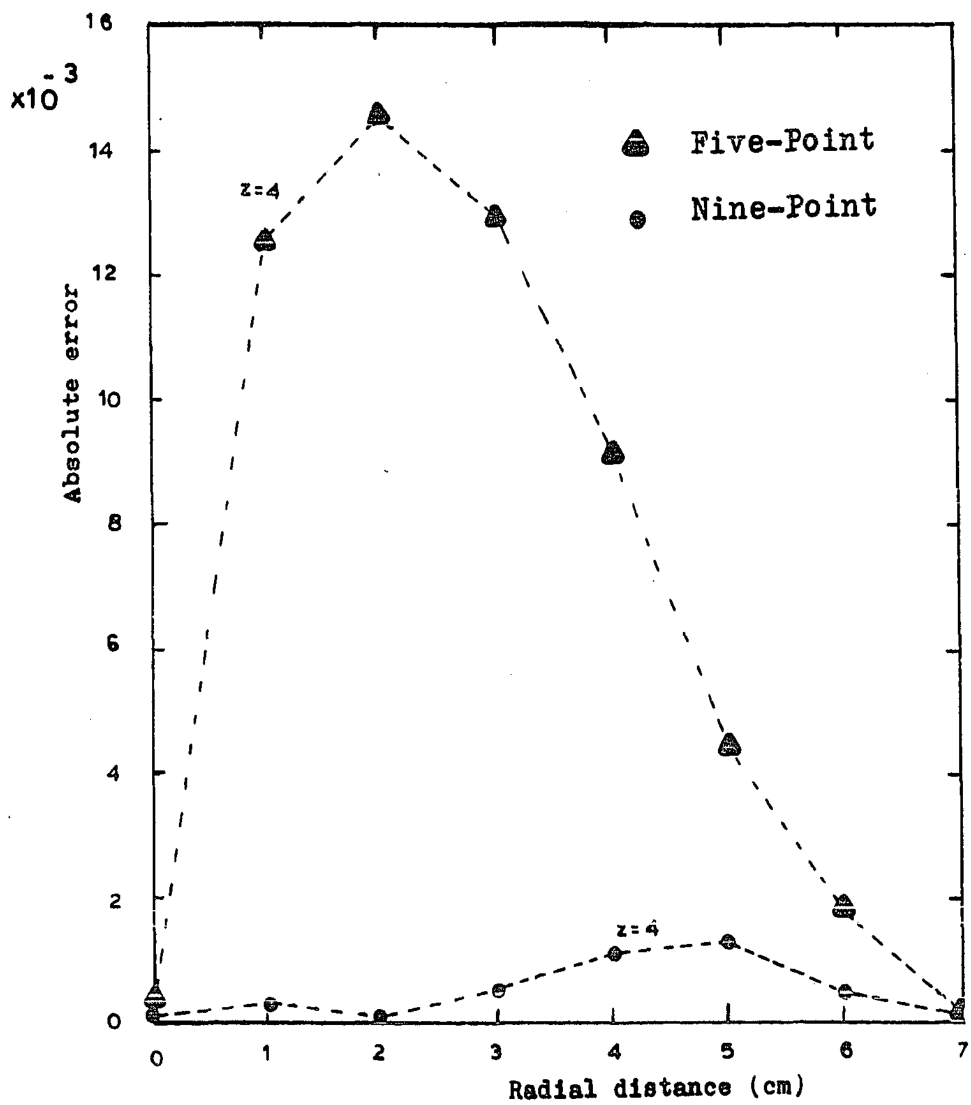


Fig. 22. Absolute error as a function of radial distance for z = 4 and equal spacing in a radially reflected cylindrical reactor core

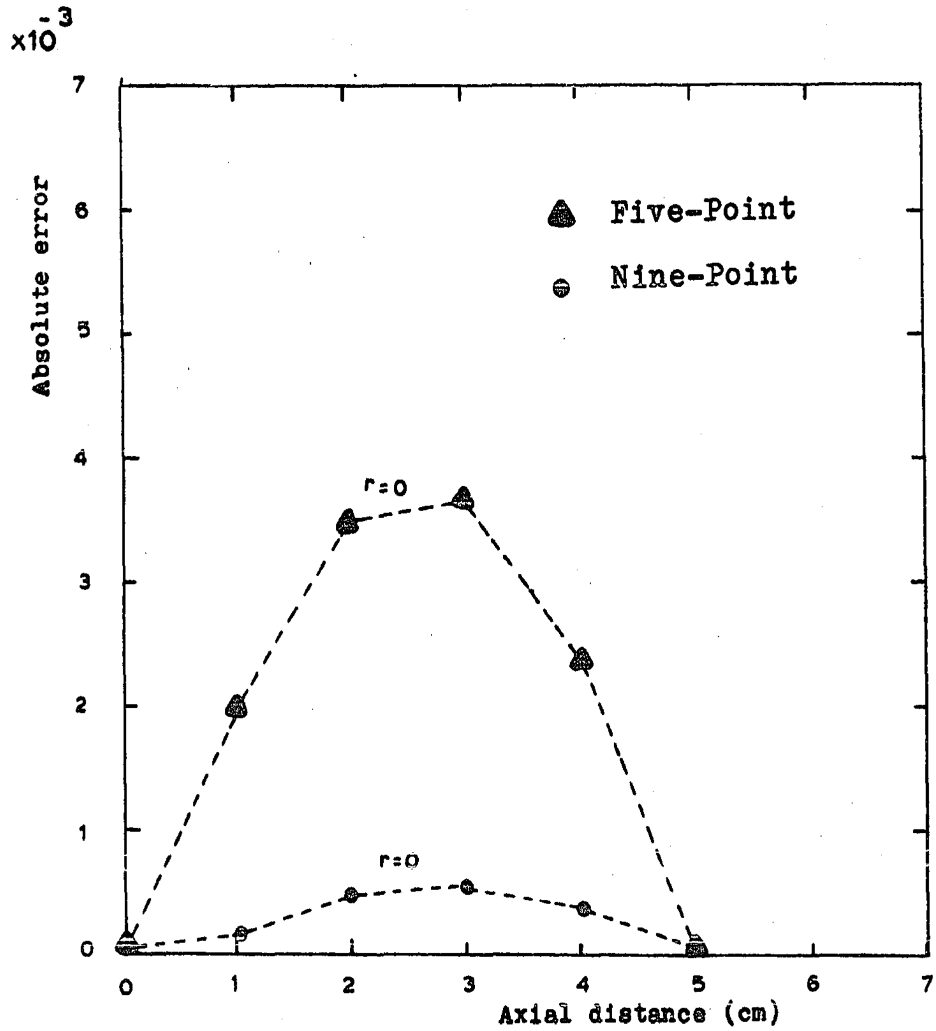


Fig. 23. Absolute error as a function of axial distance for r = 0 and equal spacing in a radially reflected cylindrical reactor core

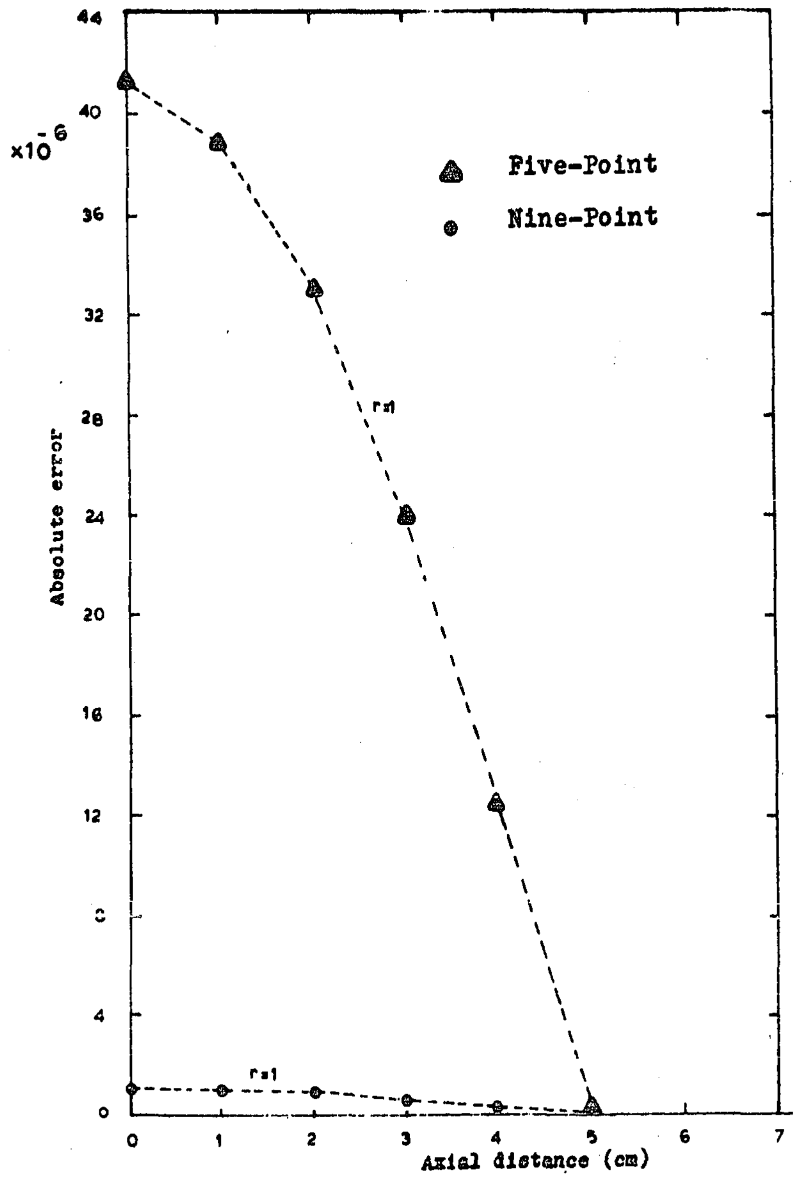


Fig. 24. Absolute error as a function of axial distance for r = 1 and equal spacing in a radially reflected cylindrical reactor core

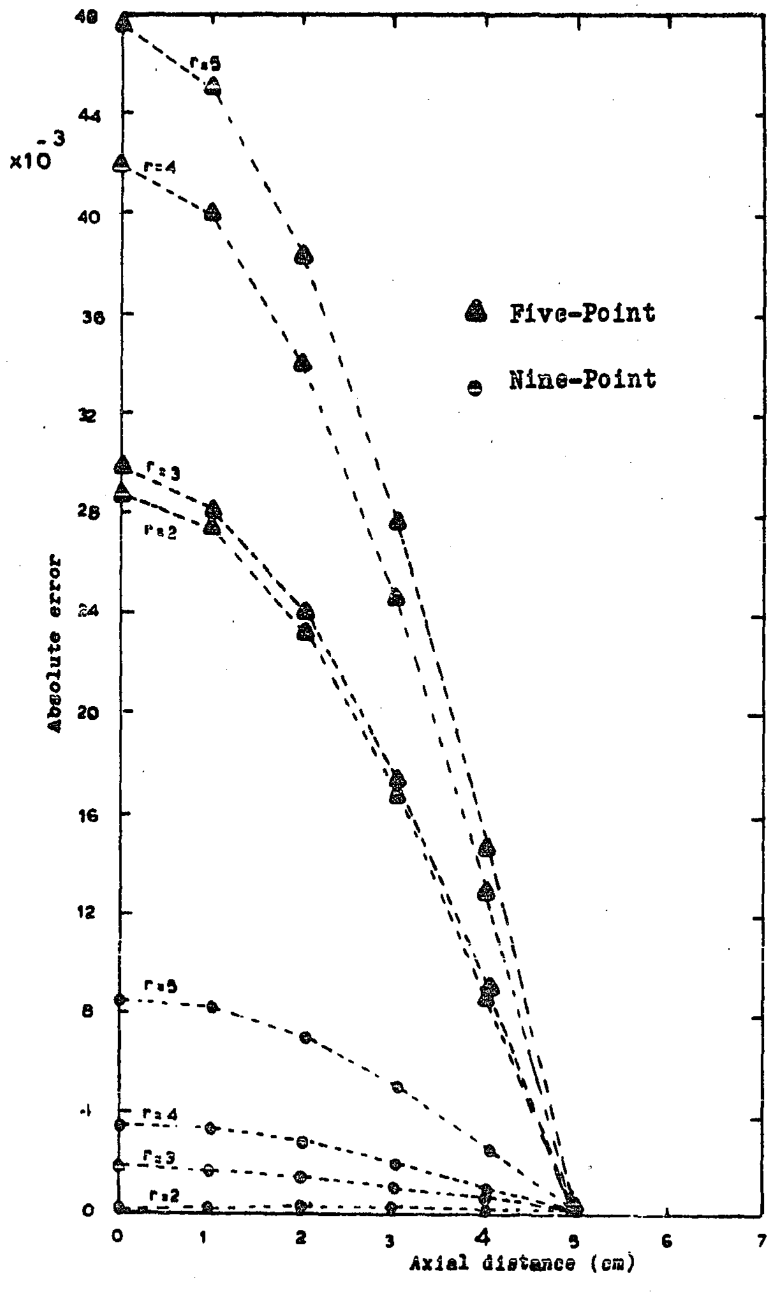


Fig. 25. Absolute error as a function of axial distance for $r = 2, 3, 4, 5$ and equal spacing radially reflected cylindrical reactor core

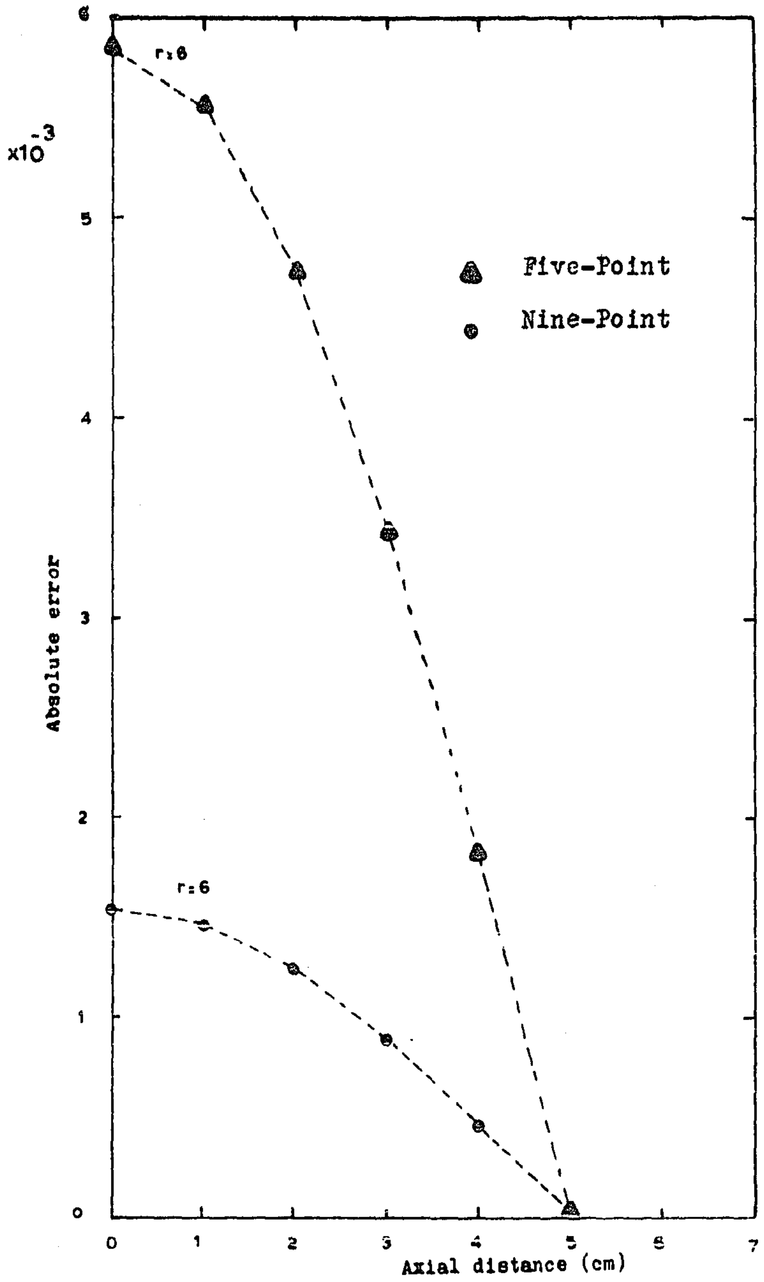


Fig. 26. Absolute error as a function of axial distance for r = 6 and equal spacing radially reflected cylindrical reactor core

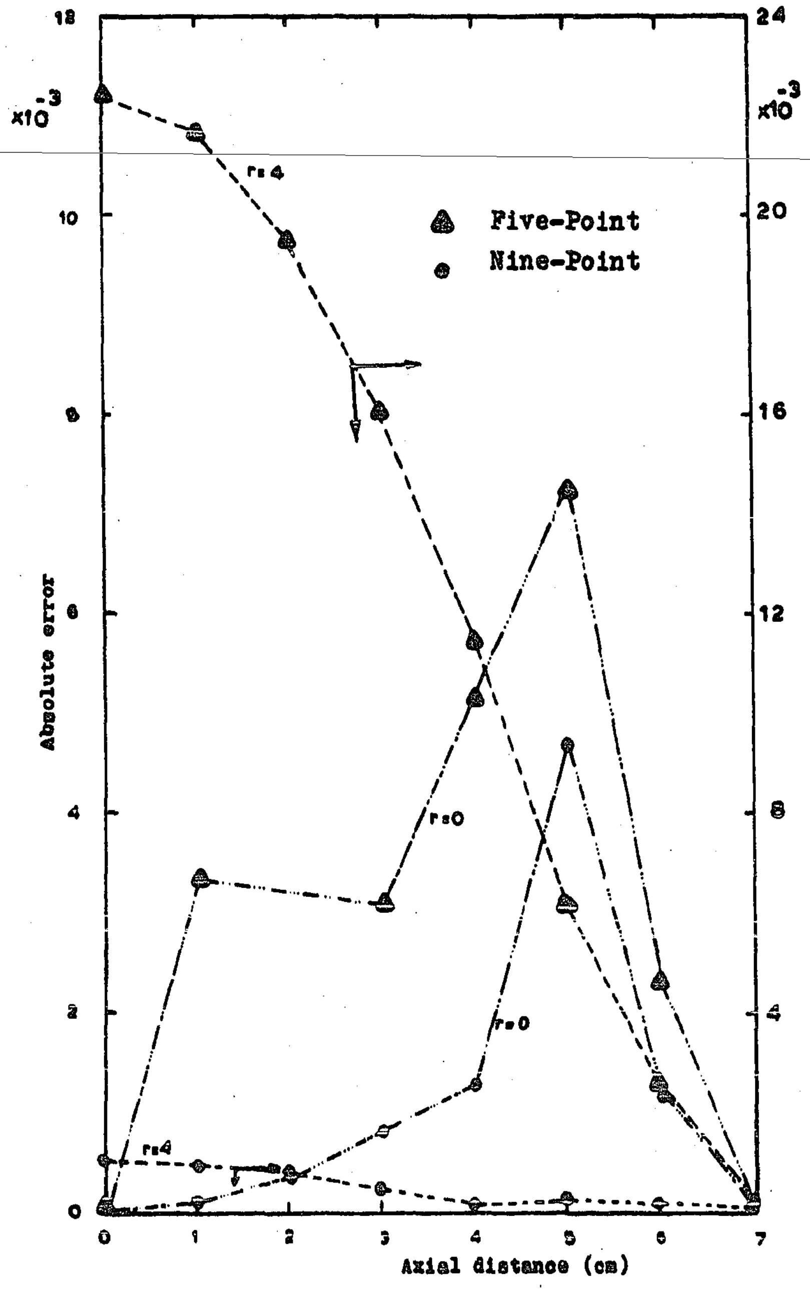


Fig. 27. Absolute error as a function of axial distance for r = 0, 4 and equal spacing in an axially reflected cylindrical reactor core

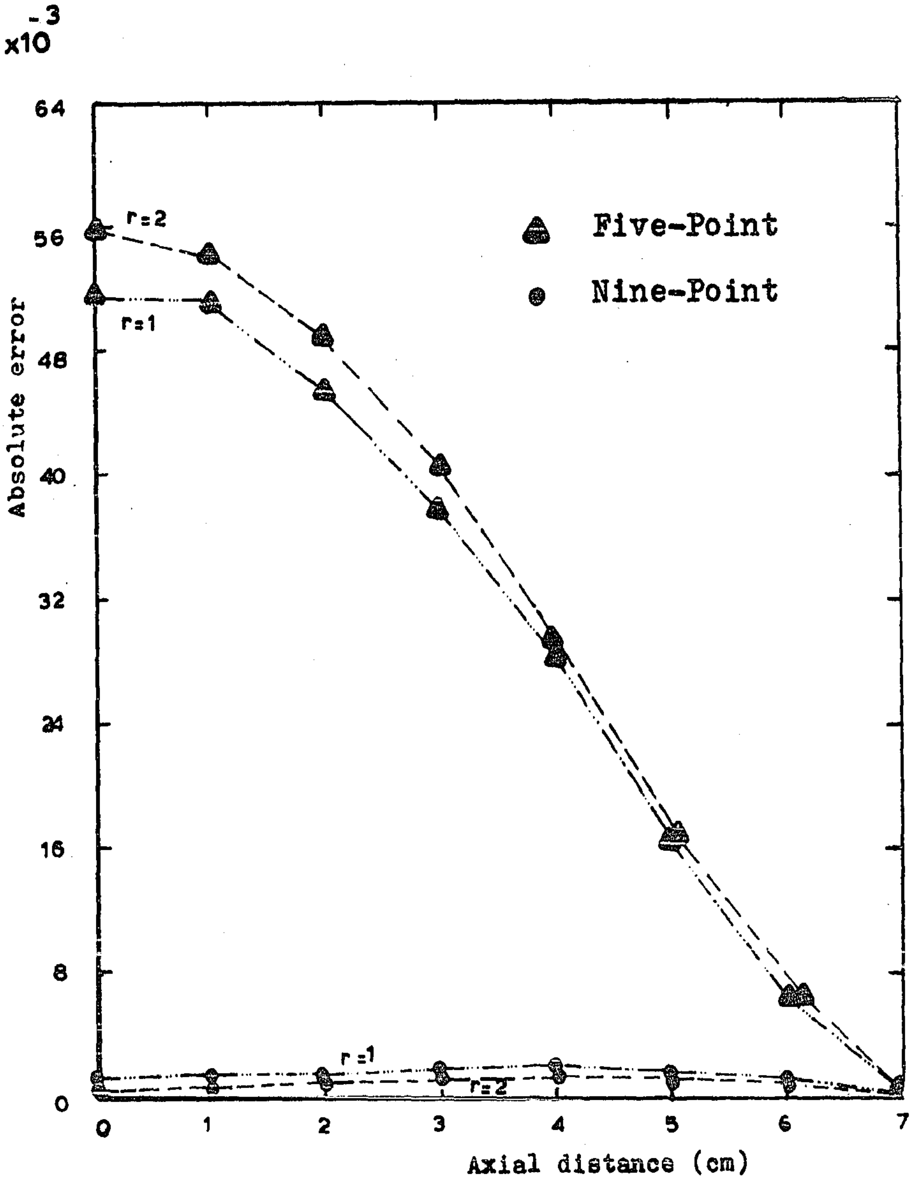


Fig. 28. Absolute error as a function of axial distance for r = 1, 2, and equal spacing axially reflected cylindrical reactor core

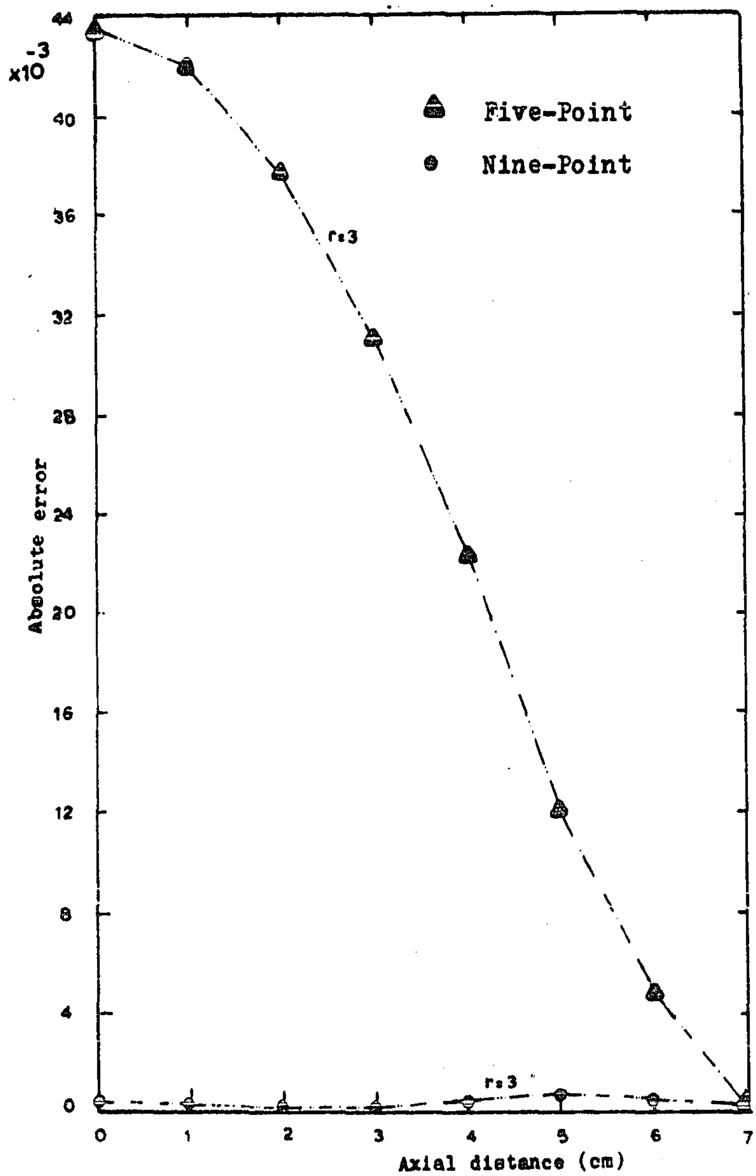


Fig. 29. Absolute error as a function of axial distance for r = 3 and equal spacing axially reflected cylindrical reactor core

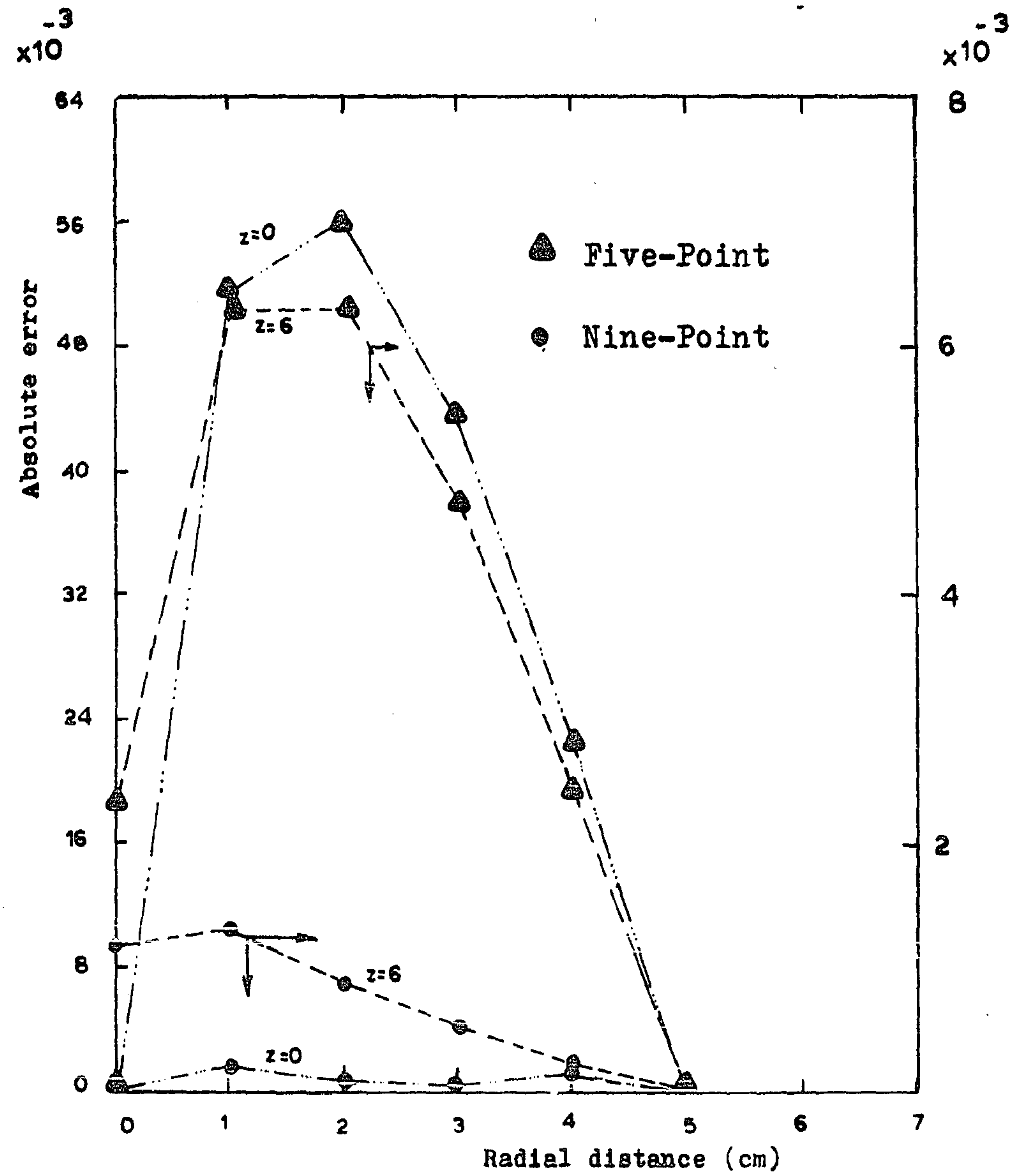


Fig. 30. Absolute error as a function of radial for z = 0, 6 and equal spacing axially reflected cylindrical reactor core

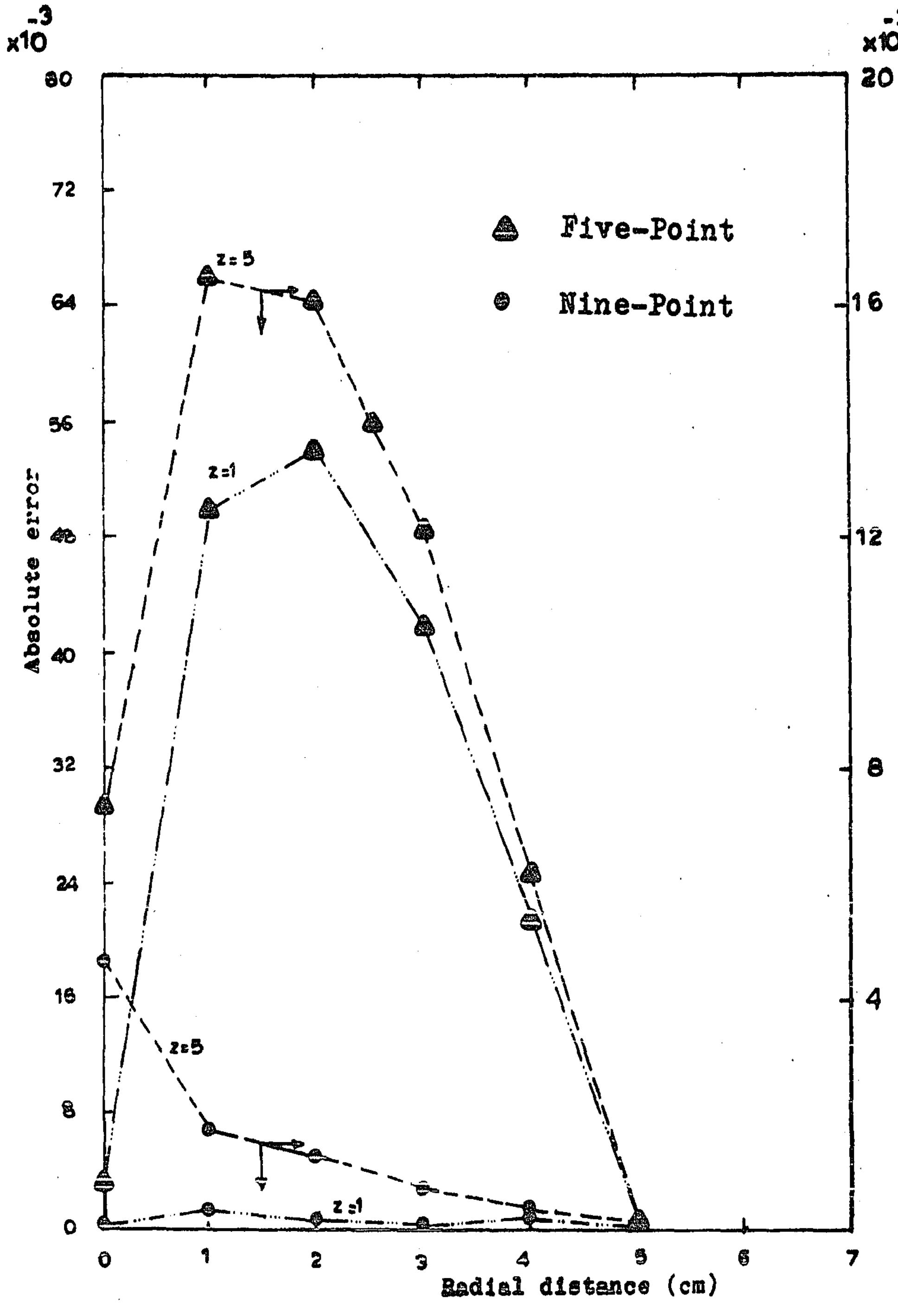


Fig. 31. Absolute error as a function of radial distance for z = 1, 5 and equal spacing axially reflected cylindrical reactor core

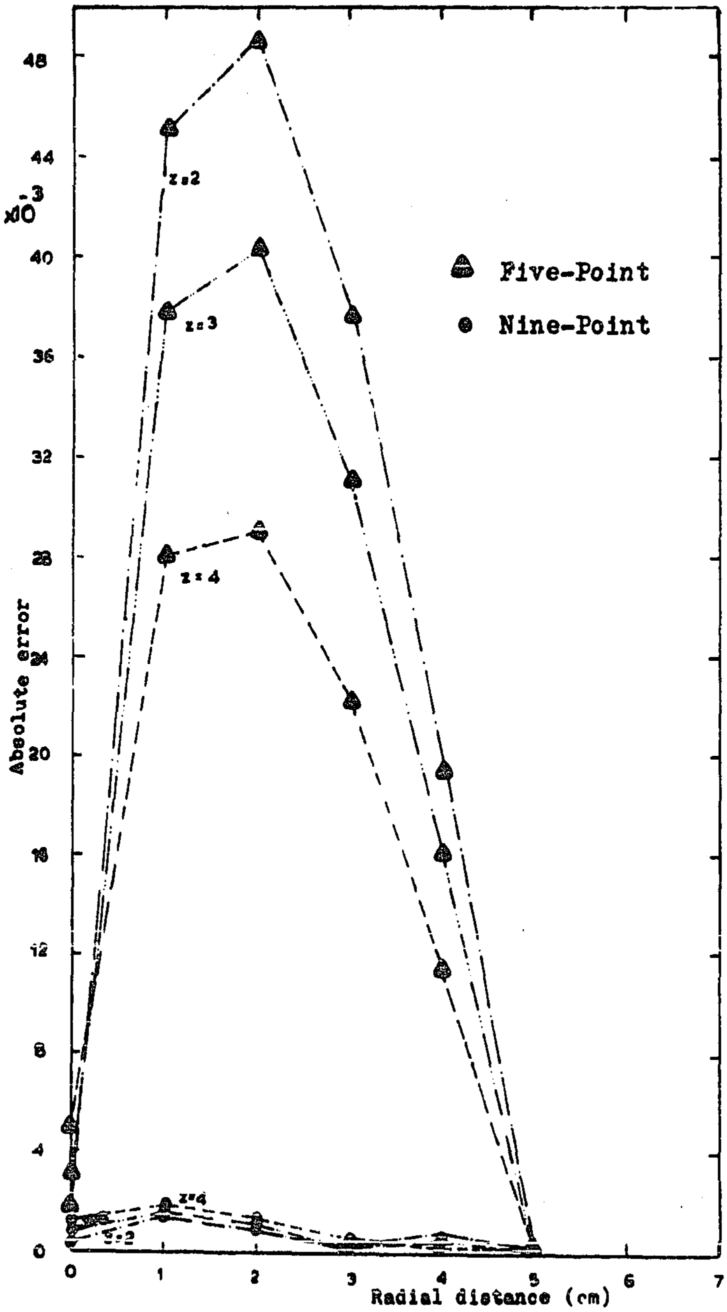


Fig. 32. Absolute error as a function of radial distance for z = 2, 3, 4 and equal spacing axially reflected cylindrical reactor core

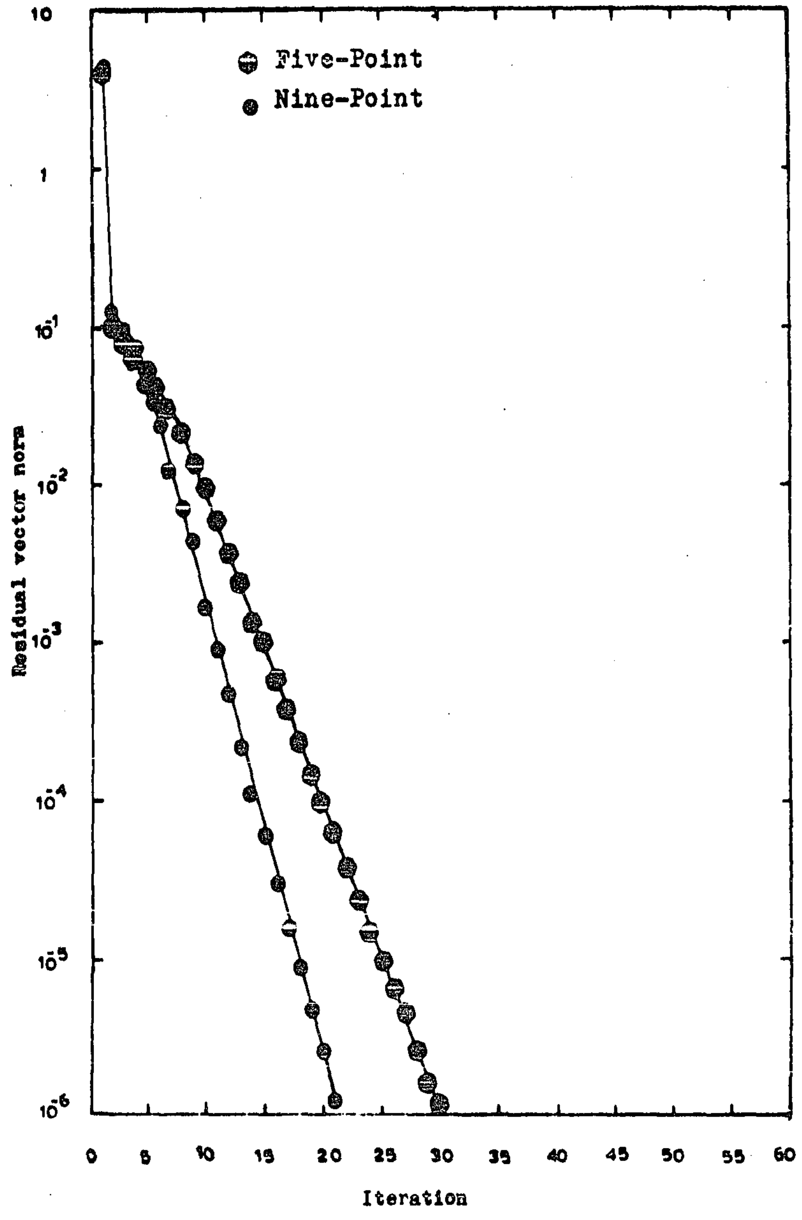


Fig. 33. Residual vector norm as a function of iteration and equal spacing in a cylindrical bare reactor core

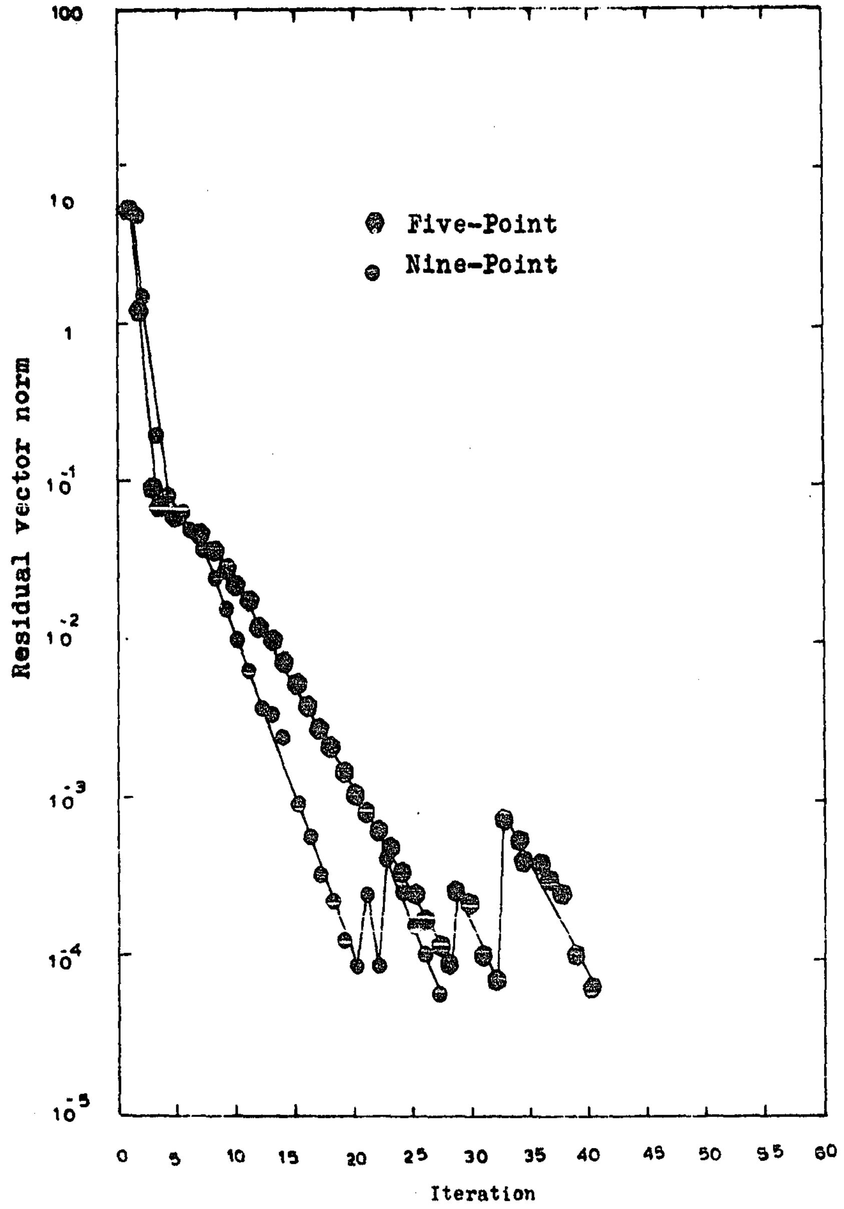


Fig. 34. Residual vector norm as a function of iteration and equal spacing in a radially reflected cylindrical reactor core

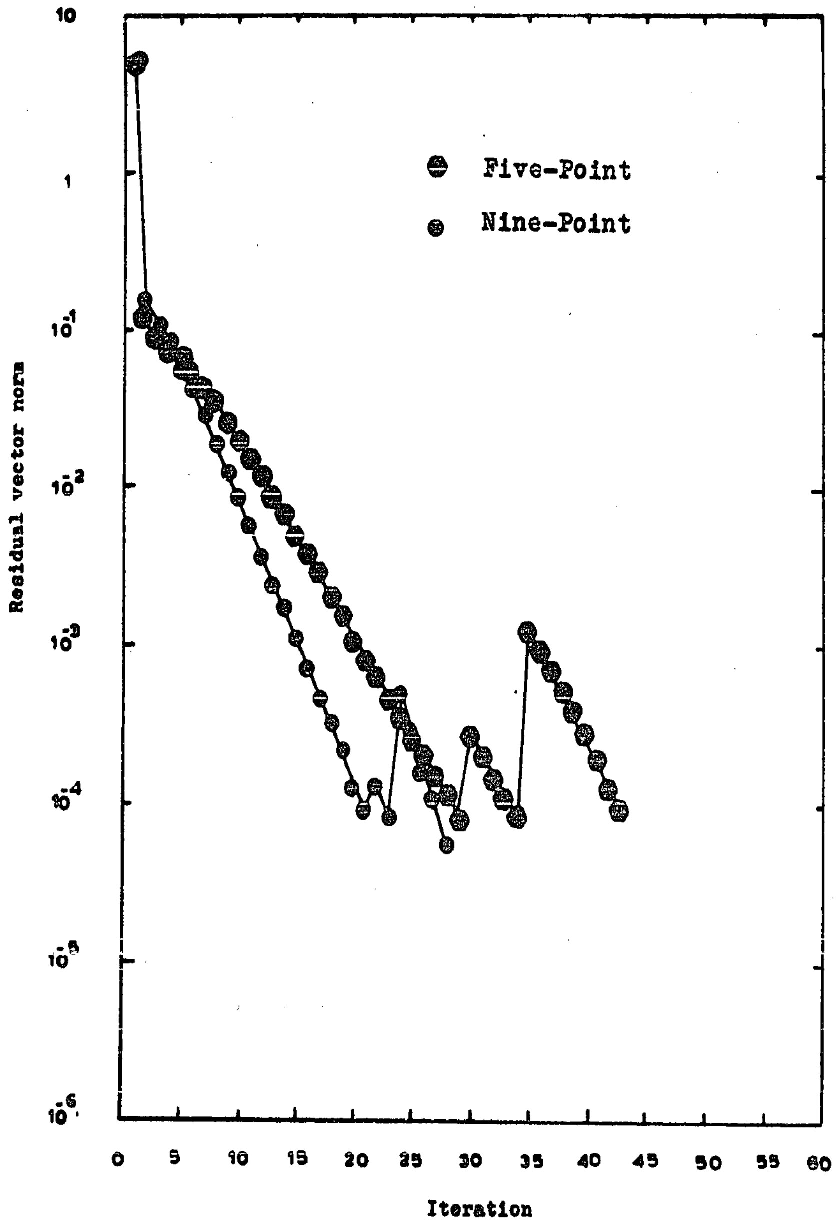


Fig. 35. Residual vector norm as a function of iteration and equal spacing in an axially reflected reactor core

Similar calculations were performed considering nonequal (Figs. 7-9) spacing along radial and axial axes for bare cylindrical cores, radially reflected cylindrical cores, and axially reflected cylindrical cores. The neutron flux values as calculated by the nine-point, the five-point, and the analytical techniques are shown in Tables 10-12. The absolute errors and Euclidean norms for the radial and axial cases are calculated and listed in Tables 13-18. Plots of absolute error versus radial and axial distances are shown in Figs. 36-53. Also, plots of the residual vector norm versus iteration are shown in Figs. 54-56.

From the plots of Figs. 36-53 one can observe the accuracy of the nine-point formulation over the five-point formulation in comparison with the analytical solution. The over all Euclidean norms of the nine-point and five-point formulas are 0.0239 and 0.0724 (bare cylindrical core), 0.032 and 0.057 (radially reflected cylindrical core), and 0.032 and 0.065 (axially reflected cylindrical core), respectively. These results indicate the accuracy of the nine-point formulation over the five-point formulation for these cases.

The convergence rate as calculated from the slopes of the curves of Figs. 54-56 for the nine-point and five-point formulas are 0.4672 and 0.3026, 0.3002 and 0.2049, and 0.349 and 0.248, respectively. These results indicate that the nine-point formula converges faster than the five-point formula.

Table 10. The flux distribution in the core for unequal spacing and for input data of $D_c$ = 1.0 cm, $\Sigma_c$ = 0.33 $cm^{-1}$, and R = z = 5 cm

| Method | z \ r | 0 | 0.5 | 1.2 | 2.1 | 3.4 | 5 |
|---|---|---|---|---|---|---|---|
| ANS | 0 | 1.000 | 0.986 | 0.919 | 0.761 | 0.435 | 0.000 |
| | 0.5 | 0.988 | 0.974 | 0.907 | 0.751 | 0.430 | 0.000 |
| | 1.2 | 0.930 | 0.916 | 0.854 | 0.707 | 0.405 | 0.000 |
| | 2.1 | 0.790 | 0.779 | 0.726 | 0.601 | 0.344 | 0.000 |
| | 3.4 | 0.482 | 0.475 | 0.443 | 0.367 | 0.210 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NPS | 0 | 1.000 | 0.987 | 0.923 | 0.769 | 0.445 | 0.000 |
| | 0.5 | 0.984 | 0.971 | 0.909 | 0.758 | 0.439 | 0.000 |
| | 1.2 | 0.924 | 0.912 | 0.854 | 0.712 | 0.412 | 0.000 |
| | 2.1 | 0.785 | 0.774 | 0.724 | 0.604 | 0.350 | 0.000 |
| | 3.4 | 0.478 | 0.471 | 0.441 | 0.367 | 0.213 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.980 | 0.920 | 0.774 | 0.453 | 0.000 |
| | 0.5 | 0.990 | 0.970 | 0.910 | 0.766 | 0.449 | 0.000 |
| | 1.2 | 0.939 | 0.920 | 0.864 | 0.727 | 0.426 | 0.000 |
| | 2.1 | 0.808 | 0.792 | 0.744 | 0.626 | 0.366 | 0.000 |
| | 3.4 | 0.501 | 0.491 | 0.461 | 0.388 | 0.227 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 11. The flux distribution in a radially reflected core for unequal spacing and for input data of $D_c$ = 0.5 cm, $\Sigma_c$ = 0.1385 $cm^{-1}$, R = z = 5 cm, $D_{R_c}$ = 1.0 cm, $\Sigma_{R_c}$ = 0.1 $cm^{-1}$, and b = 2 cm

| Method | z \ r | 0 | 0.5 | 1.2 | 2.1 | 3.4 | 5 | 5.9 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| ANS | 0 | 1.000 | 0.989 | 0.937 | 0.813 | 0.548 | 0.160 | 0.075 | 0.000 |
| | 0.5 | 0.988 | 0.977 | 0.925 | 0.803 | 0.541 | 0.159 | 0.074 | 0.000 |
| | 1.2 | 0.930 | 0.920 | 0.871 | 0.756 | 0.509 | 0.149 | 0.069 | 0.000 |
| | 2.1 | 0.790 | 0.781 | 0.740 | 0.642 | 0.433 | 0.127 | 0.059 | 0.000 |
| | 3.4 | 0.482 | 0.476 | 0.451 | 0.392 | 0.264 | 0.077 | 0.036 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NPS | 0 | 1.000 | 0.988 | 0.938 | 0.817 | 0.556 | 0.168 | 0.078 | 0.000 |
| | 0.5 | 0.984 | 0.972 | 0.922 | 0.803 | 0.546 | 0.165 | 0.077 | 0.000 |
| | 1.2 | 0.922 | 0.911 | 0.864 | 0.752 | 0.512 | 0.155 | 0.072 | 0.000 |
| | 2.1 | 0.781 | 0.771 | 0.731 | 0.636 | 0.433 | 0.131 | 0.061 | 0.000 |
| | 3.4 | 0.475 | 0.468 | 0.444 | 0.386 | 0.263 | 0.080 | 0.037 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.979 | 0.925 | 0.803 | 0.541 | 0.151 | 0.070 | 0.000 |
| | 0.5 | 0.984 | 0.964 | 0.911 | 0.791 | 0.532 | 0.149 | 0.069 | 0.000 |
| | 1.2 | 0.924 | 0.904 | 0.855 | 0.742 | 0.499 | 0.139 | 0.065 | 0.000 |
| | 2.1 | 0.785 | 0.768 | 0.726 | 0.630 | 0.424 | 0.118 | 0.055 | 0.000 |
| | 3.4 | 0.480 | 0.469 | 0.443 | 0.385 | 0.259 | 0.072 | 0.034 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 12. The flux distribution in the axially reflected core for unequal spacing and input data of $D_c$ = 0.5 cm, $\Sigma_c$ = 0.1536 $cm^{-1}$, R = z = 5 cm, $D_e$ = 1.0 cm, $\Sigma_e$ = 0.1, and b = 2 cm

| Method | r \ z | 0 | 0.5 | 1.2 | 2.1 | 3.4 | 5 | 5.9 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| ANS | 0 | 1.000 | 0.991 | 0.946 | 0.837 | 0.593 | 0.192 | 0.091 | 0.000 |
| | 0.5 | 0.986 | 0.976 | 0.932 | 0.825 | 0.584 | 0.189 | 0.090 | 0.000 |
| | 1.2 | 0.918 | 0.910 | 0.867 | 0.769 | 0.544 | 0.175 | 0.084 | 0.000 |
| | 2.1 | 0.761 | 0.754 | 0.720 | 0.637 | 0.451 | 0.146 | 0.070 | 0.000 |
| | 3.4 | 0.435 | 0.431 | 0.412 | 0.364 | 0.258 | 0.084 | 0.040 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NPS | 0 | 1.000 | 0.987 | 0.939 | 0.828 | 0.585 | 0.184 | 0.089 | 0.000 |
| | 0.5 | 0.985 | 0.972 | 0.924 | 0.815 | 0.574 | 0.184 | 0.087 | 0.000 |
| | 1.2 | 0.918 | 0.907 | 0.862 | 0.760 | 0.535 | 0.172 | 0.081 | 0.000 |
| | 2.1 | 0.763 | 0.753 | 0.716 | 0.631 | 0.444 | 0.143 | 0.068 | 0.000 |
| | 3.4 | 0.440 | 0.435 | 0.413 | 0.364 | 0.256 | 0.083 | 0.039 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.988 | 0.940 | 0.829 | 0.583 | 0.181 | 0.086 | 0.000 |
| | 0.5 | 0.973 | 0.961 | 0.915 | 0.807 | 0.567 | 0.176 | 0.084 | 0.000 |
| | 1.2 | 0.905 | 0.894 | 0.851 | 0.750 | 0.527 | 0.164 | 0.078 | 0.000 |
| | 2.1 | 0.754 | 0.744 | 0.708 | 0.625 | 0.439 | 0.136 | 0.065 | 0.000 |
| | 3.4 | 0.437 | 0.431 | 0.410 | 0.362 | 0.254 | 0.079 | 0.038 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 13. The calculated results of absolute errors and Euclidean norms for unequal radial spacing

| z | r | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.014 | 0.0239 | 0.0000 | 0.0152 | 0.0729 |
| | 0.5 | 0.9925 | | | 5.4548 | | |
| | 1.2 | 4.4811 | | | 1.5076 | | |
| | 2.1 | 8.4843 | | | 13.3880 | | |
| | 3.4 | 10.1004 | | | 17.0610 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 0.5 | 0 | 3.4117 | 0.0116 | | 1.4338 | 0.0244 | |
| | 0.5 | 2.0372 | | | 3.4939 | | |
| | 1.2 | 1.7118 | | | 3.2665 | | |
| | 2.1 | 6.2204 | | | 14.7191 | | |
| | 3.4 | 8.7662 | | | 18.7148 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1.2 | 0 | 5.3882 | 0.0108 | | 8.9705 | 0.0316 | |
| | 0.5 | 4.1133 | | | 3.7182 | | |
| | 1.2 | 0.4191 | | | 9.6519 | | |
| | 2.1 | 4.2172 | | | 19.3912 | | |
| | 3.4 | 7.3282 | | | 20.8571 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 2.1 | 0 | 5.0212 | 0.0093 | | 18.1284 | 0.0405 | |
| | 0.5 | 4.5887 | | | 13.4563 | | |
| | 1.2 | 1.5651 | | | 17.8663 | | |
| | 2.1 | 2.5360 | | | 24.6107 | | |
| | 3.4 | 5.5770 | | | 22.4860 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3.4 | 0 | 3.4133 | 0.0061 | | 19.5374 | 0.0423 | |
| | 0.5 | 3.4992 | | | 16.5201 | | |
| | 1.2 | 1.7094 | | | 18.9820 | | |
| | 2.1 | 0.9048 | | | 21.5734 | | |
| | 3.4 | 2.9908 | | | 17.5557 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 14. The calculated results of absolute errors and Euclidean norms for unequal axial spacing

| r | z | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.014 | 0.0239 | 0.0000 | 0.015 | 0.0729 |
| | 0.5 | 3.4117 | | | 1.9705 | | |
| | 1.2 | 5.3882 | | | 8.9705 | | |
| | 2.1 | 5.0212 | | | 18.1284 | | |
| | 3.4 | 3.4133 | | | 19.5374 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 0.5 | 0 | 0.9925 | 0.007 | | 5.4548 | 0.023 | |
| | 0.5 | 2.0372 | | | 3.4939 | | |
| | 1.2 | 4.1133 | | | 3.7182 | | |
| | 2.1 | 4.5887 | | | 13.4563 | | |
| | 3.4 | 3.4992 | | | 16.5201 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1.2 | 0 | 4.4811 | 0.005 | | 1.5076 | 0.028 | |
| | 0.5 | 1.7118 | | | 3.2665 | | |
| | 1.2 | 0.4191 | | | 9.6519 | | |
| | 2.1 | 1.5651 | | | 17.8663 | | |
| | 3.4 | 1.7094 | | | 18.982 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 2.1 | 0 | 8.4843 | 0.012 | | 13.3880 | 0.043 | |
| | 0.5 | 6.2204 | | | 14.7191 | | |
| | 1.2 | 4.2172 | | | 19.3912 | | |
| | 2.1 | 2.5360 | | | 24.6107 | | |
| | 3.4 | 0.9048 | | | 21.5734 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3.4 | 0 | 10.1004 | 0.017 | | 17.0610 | 0.044 | |
| | 0.5 | 8.7662 | | | 18.7148 | | |
| | 1.2 | 7.3282 | | | 20.8571 | | |
| | 2.1 | 5.770 | | | 22.4860 | | |
| | 3.4 | 2.9900 | | | 17.5557 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 15. The calculated results for the absolute errors and Euclidean norms for unequal spacing in a radially reflected core at different axial levels

| z | r | Nine-point | | | Five-point | | |
|---|---|---|---|---|---|---|---|
| | | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm |
| 0 | 0 | 0.0000 | 0.012 | 0.032 | 0.0000 | 0.022 | 0.057 |
| | 0.5 | 5.5330 | | | 9.8640 | | |
| | 1.2 | 8.3670 | | | 11.6643 | | |
| | 2.1 | 3.7412 | | | 9.5740 | | |
| | 3.4 | 8.0554 | | | 7.0512 | | |
| | 5 | 7.6262 | | | 9.4980 | | |
| | 5.9 | 3.4505 | | | 4.5772 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 0.5 | 0 | 4.1068 | 0.017 | | 3.5372 | 0.028 | |
| | 0.5 | 4.6682 | | | 13.2392 | | |
| | 1.2 | 3.1232 | | | 14.8633 | | |
| | 2.1 | 0.2216 | | | 12.3833 | | |
| | 3.4 | 5.5649 | | | 8.9519 | | |
| | 5 | 6.8353 | | | 9.9392 | | |
| | 5.9 | 3.0811 | | | 4.7800 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 1.2 | 0 | 8.0393 | 0.016 | | 5.9306 | 0.031 | |
| | 0.5 | 8.6475 | | | 15.0606 | | |
| | 1.2 | 7.1029 | | | 16.4963 | | |
| | 2.1 | 3.512 | | | 13.8742 | | |

Table 15 (Continued)

| z | r | Nine-point $E_i \times 10^{-3}$ | Nine-point $\ell_2$ | Nine-point Over all norm | Five-point $E_i \times 10^{-3}$ | Five-point $\ell_2$ | Five-point Over all norm |
|---|---|---|---|---|---|---|---|
| | 3.4 | 2.6604 | | | 9.9463 | | |
| | 5 | 5.7403 | | | 9.7861 | | |
| | 5.9 | 2.5744 | | | 4.6997 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 2.1 | 0 | 9.5645 | 0.019 | | 5.5013 | 0.027 | |
| | 0.5 | 10.6341 | | | 13.2954 | | |
| | 1.2 | 9.4261 | | | 14.5336 | | |
| | 2.1 | 6.0955 | | | 12.2742 | | |
| | 3.4 | 0.0501 | | | 8.8018 | | |
| | 5 | 4.4338 | | | 8.4194 | | |
| | 5.9 | 1.9461 | | | 4.0422 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 3.4 | 0 | 6.9750 | 0.0145 | | 2.4525 | 0.0150 | |
| | 0.5 | 8.1708 | | | 7.2482 | | |
| | 1.2 | 7.6274 | | | 8.0739 | | |
| | 2.1 | 5.5059 | | | 6.8192 | | |
| | 3.4 | 1.2635 | | | 4.9316 | | |
| | 5 | 2.4241 | | | 5.0145 | | |
| | 5.9 | 1.0739 | | | 2.4101 | | |
| | 7 | 0.0000 | | | 0.0000 | | |

Table 16. The calculated results for absolute errors and Euclidean norms for unequal spacing in a radially reflected core at different radial levels

| r | z | Nine-point | | | Five-point | | |
|---|---|---|---|---|---|---|---|
| | | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm |
| 0 | 0 | 0.0000 | 0.015 | 0.032 | 0.0000 | 0.012 | 0.057 |
| | 0.5 | 4.1068 | | | 3.5372 | | |
| | 1.2 | 8.0393 | | | 5.9306 | | |
| | 2.1 | 9.5645 | | | 5.5013 | | |
| | 3.4 | 6.9750 | | | 2.4525 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 0.5 | 0 | 5.5330 | 0.014 | | 9.8639 | 0.027 | |
| | 0.5 | 4.66882 | | | 13.2392 | | |
| | 1.2 | 8.6475 | | | 15.0606 | | |
| | 2.1 | 10.6341 | | | 13.2954 | | |
| | 3.4 | 8.1708 | | | 7.2482 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1.2 | 0 | 0.8367 | 0.014 | | 11.6693 | 0.030 | |
| | 0.5 | 3.1232 | | | 14.8633 | | |
| | 1.2 | 7.1029 | | | 16.4963 | | |
| | 2.1 | 9.4261 | | | 14.5336 | | |
| | 3.4 | 7.6274 | | | 8.0739 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3.4 | 0 | 8.0554 | 0.010 | | 7.0512 | 0.018 | |
| | 0.5 | 5.5649 | | | 8.9519 | | |
| | 1.2 | 2.6604 | | | 9.9463 | | |

Table 16 (Continued)

| r | z | Nine-point | | | Five-point | | |
|---|---|---|---|---|---|---|---|
| | | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm |
| | 2.1 | 0.0501 | | | 8.8018 | | |
| | 3.4 | 1.2635 | | | 4.9316 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 5 | 0 | 7.6262 | 0.013 | | 9.498 | 0.020 | |
| | 0.5 | 6.8353 | | | 9.9392 | | |
| | 1.2 | 5.7403 | | | 9.7861 | | |
| | 2.1 | 4.4338 | | | 8.4194 | | |
| | 3.4 | 2.4241 | | | 5.0145 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 5.9 | 0 | 3.4505 | 0.006 | | 4.5772 | 0.009 | |
| | 0.5 | 3.0811 | | | 4.7800 | | |
| | 1.2 | 2.5744 | | | 4.6997 | | |
| | 2.1 | 1.9461 | | | 4.0422 | | |
| | 3.4 | 1.0739 | | | 2.4101 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

Table 17. The calculated results for the absolute errors and Euclidean norms for unequal spacing in an axially reflected core at different radial levels

| r | z | Nine-point | | | Five-point | | |
|---|---|---|---|---|---|---|---|
| | | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm |
| 0 | 0 | 0.0000 | 0.017 | 0.032 | 0.0000 | 0.019 | 0.065 |
| | 0.5 | 3.3114 | | | 3.0821 | | |
| | 1.2 | 6.8140 | | | 6.0966 | | |
| | 2.1 | 9.1620 | | | 8.1434 | | |
| | 3.4 | 7.7172 | | | 9.7077 | | |
| | 5 | 8.0772 | | | 11.3705 | | |
| | 5.9 | 2.4637 | | | 5.3258 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 0.5 | 0 | 1.0691 | 0.018 | | 12.3577 | 0.039 | |
| | 0.5 | 4.4104 | | | 15.2753 | | |
| | 1.2 | 7.9860 | | | 17.7134 | | |
| | 2.1 | 10.4298 | | | 18.4130 | | |
| | 3.4 | 10.0221 | | | 16.9178 | | |
| | 5 | 5.8732 | | | 13.5052 | | |
| | 5.9 | 2.9078 | | | 6.3452 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 1.2 | 0 | 0.0562 | 0.016 | | 13.1323 | 0.040 | |
| | 0.5 | 3.2333 | | | 15.8654 | | |
| | 1.2 | 6.7322 | | | 18.1159 | | |
| | 2.1 | 9.3664 | | | 18.8638 | | |

Table 17 (Continued)

| r | z | Nine-point | | | Five-point | | |
|---|---|---|---|---|---|---|---|
| | | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm |
| | 3.4 | 9.4447 | | | 16.8525 | | |
| | 5 | 4.6853 | | | 12.9399 | | |
| | 5.9 | 2.5704 | | | 6.0842 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 2.1 | 0 | 2.2877 | 0.011 | | 6.8954 | 0.026 | |
| | 0.5 | 0.4168 | | | 9.2383 | | |
| | 1.2 | 3.5298 | | | 11.3365 | | |
| | 2.1 | 6.1349 | | | 12.2482 | | |
| | 3.4 | 6.8002 | | | 11.7533 | | |
| | 5 | 3.3133 | | | 10.0476 | | |
| | 5.9 | 1.9236 | | | 4.7225 | | |
| | 7 | 0.0000 | | | 0.0000 | | |
| 3.4 | 0 | 5.0542 | 0.0067 | | 1.3631 | 0.0073 | |
| | 0.5 | 3.4285 | | | 0.0595 | | |
| | 1.2 | 1.4008 | | | 1.5368 | | |
| | 2.1 | 0.6080 | | | 2.6653 | | |
| | 3.4 | 1.9292 | | | 3.6903 | | |
| | 5 | 1.1605 | | | 4.8132 | | |
| | 5.9 | 0.8743 | | | 2.2579 | | |
| | 7 | 0.0000 | | | 0.0000 | | |

Table 18. The calculated results for the absolute errors and Euclidean norms for unequal spacing in an axially reflected core at different axial levels

| z | r | Nine-point | | | Five-point | | |
|---|---|---|---|---|---|---|---|
| | | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm |
| 0 | 0 | 0.0000 | 0.006 | 0.032 | 0.0000 | 0.019 | 0.065 |
| | 0.5 | 1.0691 | | | 12.3577 | | |
| | 1.2 | 0.0562 | | | 13.1323 | | |
| | 2.1 | 2.2877 | | | 6.8954 | | |
| | 3.4 | 5.0542 | | | 1.3631 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 0.5 | 0 | 3.3114 | 0.007 | | 3.0821 | 0.025 | |
| | 0.5 | 4.4104 | | | 15.2753 | | |
| | 1.2 | 3.2333 | | | 15.8654 | | |
| | 2.1 | 0.4168 | | | 9.2383 | | |
| | 3.4 | 3.4285 | | | 0.0595 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 1.2 | 0 | 6.8140 | 0.013 | | 6.0966 | 0.029 | |
| | 0.5 | 7.9860 | | | 17.7134 | | |
| | 1.2 | 6.7322 | | | 18.1159 | | |
| | 2.1 | 3.5298 | | | 11.3365 | | |
| | 3.4 | 1.4008 | | | 1.5368 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 2.1 | 0 | 9.1620 | 0.018 | | 8.1434 | 0.030 | |
| | 0.5 | 10.4298 | | | 18.4130 | | |
| | 1.2 | 9.3664 | | | 18.8638 | | |

Table 18 (Continued)

| z | r | Nine-point | | | Five-point | | |
|---|---|---|---|---|---|---|---|
| | | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm | $E_i \times 10^{-3}$ | $\ell_2$ | Over all norm |
| | 2.1 | 6.1349 | | | 12.2482 | | |
| | 3.4 | 0.6080 | | | 2.6653 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 3.4 | 0 | 7.7172 | 0.017 | | 9.7077 | 0.029 | |
| | 0.5 | 10.0221 | | | 16.9178 | | |
| | 1.2 | 9.4447 | | | 16.8525 | | |
| | 2.1 | 6.8002 | | | 11.7533 | | |
| | 3.4 | 1.9292 | | | 3.6903 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 5 | 0 | 8.0772 | 0.012 | | 11.3705 | 0.025 | |
| | 0.5 | 5.8732 | | | 13.5052 | | |
| | 1.2 | 4.6853 | | | 12.9349 | | |
| | 2.1 | 3.3133 | | | 10.0476 | | |
| | 3.4 | 1.1605 | | | 4.8132 | | |
| | 5 | 0.0000 | | | 0.0000 | | |
| 5.9 | 0 | 2.4637 | 0.005 | | 5.3258 | 0.012 | |
| | 0.5 | 2.9078 | | | 6.3452 | | |
| | 1.2 | 2.5704 | | | 6.0842 | | |
| | 2.1 | 1.9236 | | | 4.7225 | | |
| | 3.4 | 0.8743 | | | 2.2579 | | |
| | 5 | 0.0000 | | | 0.0000 | | |

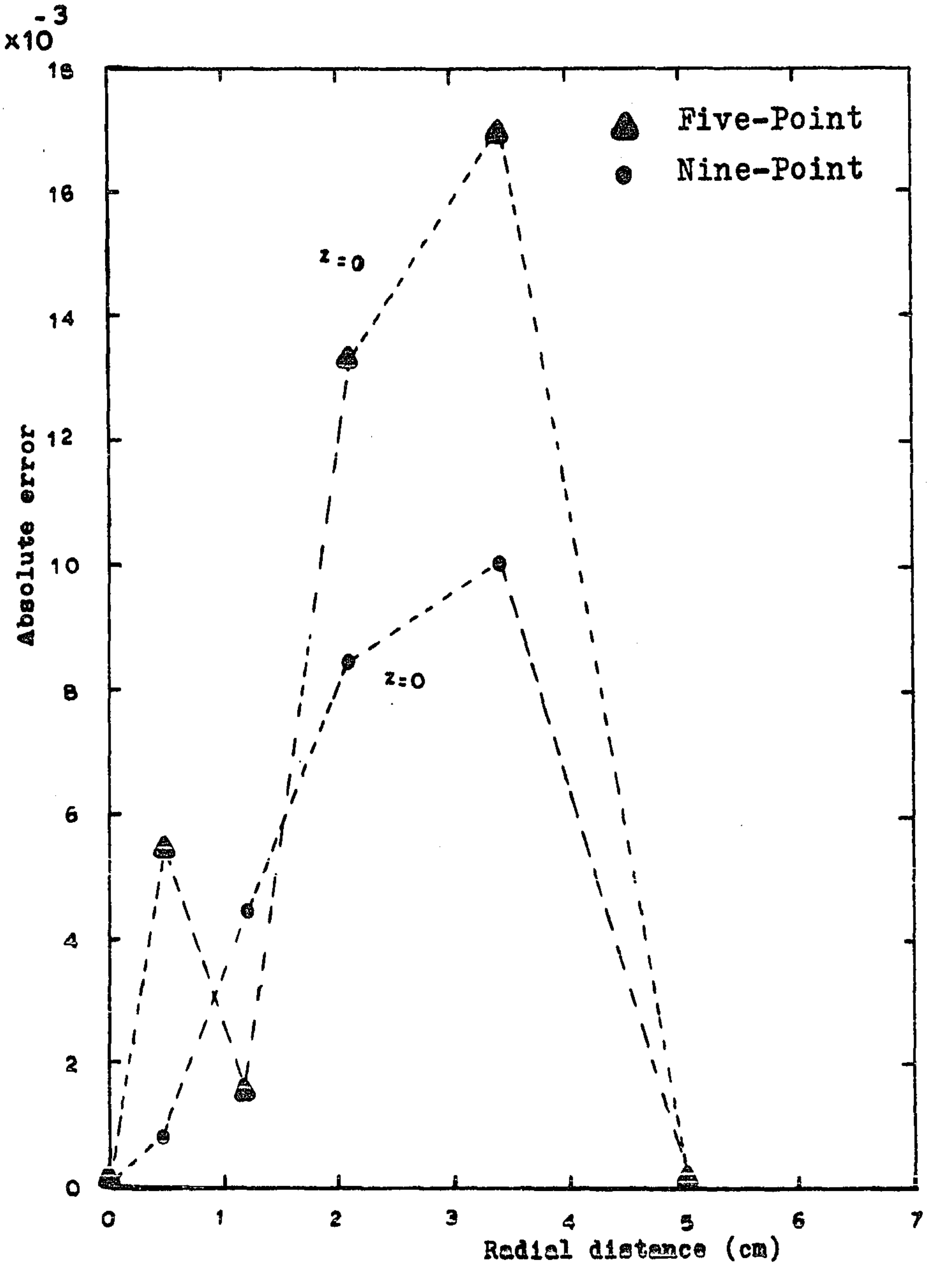


Fig. 36. Absolute error as a function radial distance for z = 0 and unequal spacing in a cylindrical bare reactor core

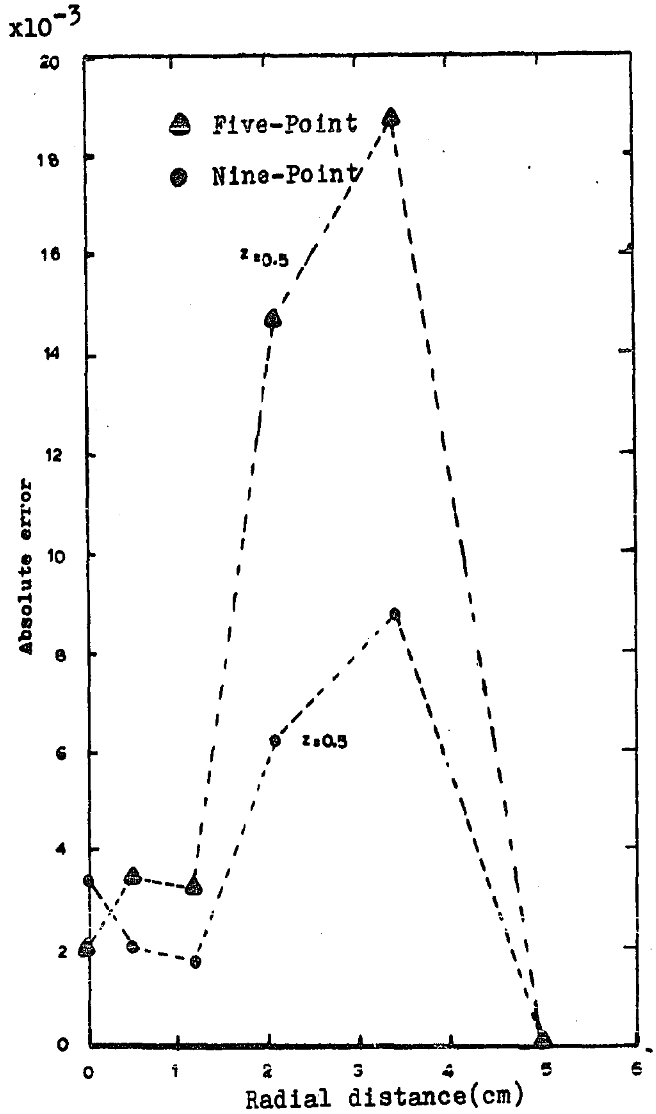


Fig. 37. Absolute error as a function of radial distance for z = 0.5 and unequal spacing cylindrical bare reactor core

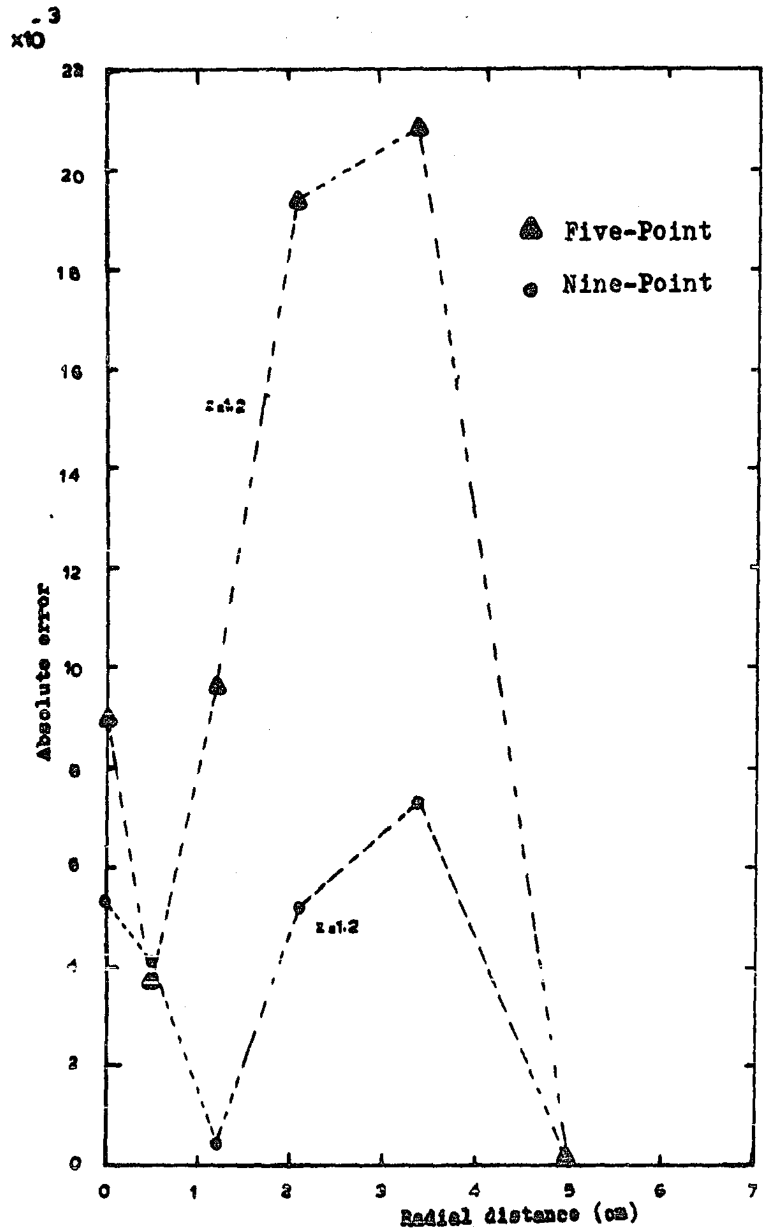


Fig. 38. Absolute error as a function of radial distance for z = 1.2 and unequal spacing cylindrical bare reactor core

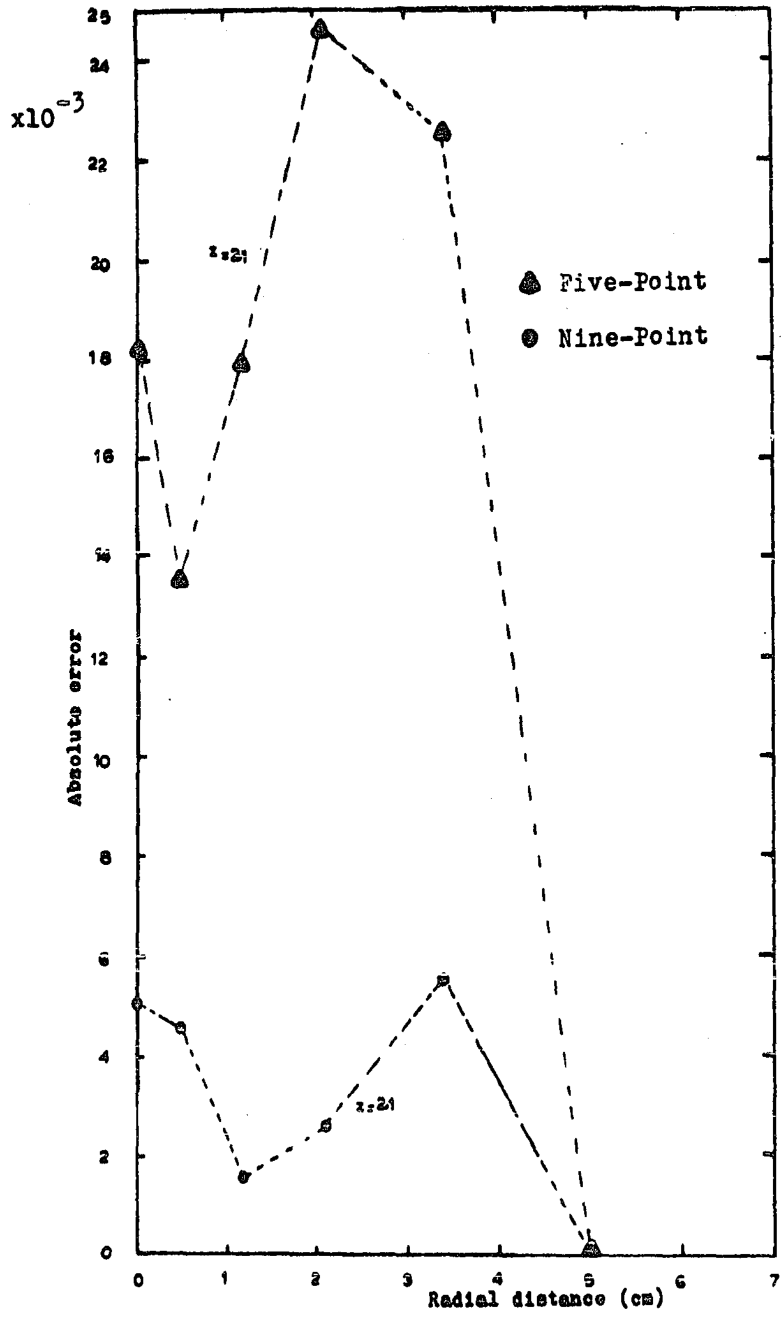


Fig. 39. Absolute error as a function of radial distance for z = 2.1 and unequal spacing cylindrical bare reactor core

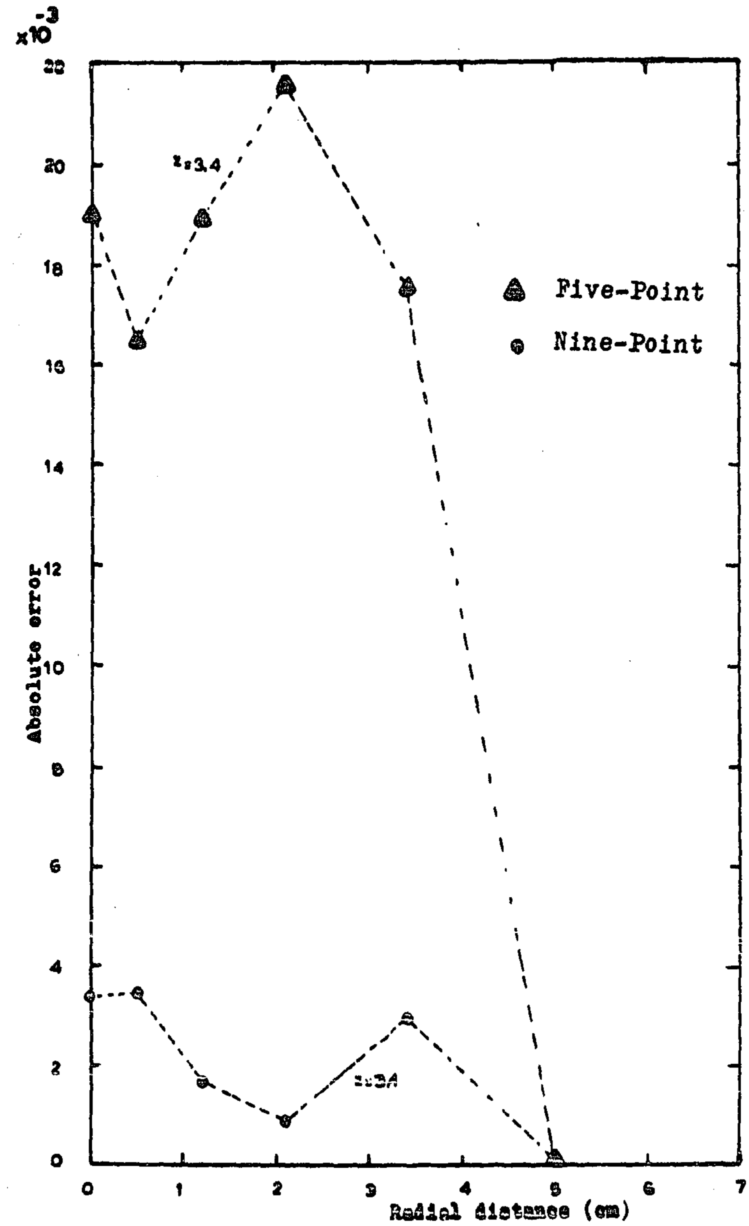


Fig. 40. Absolute error as a function of radial distance for z = 3.4 and unequal spacing cylindrical bare reactor core

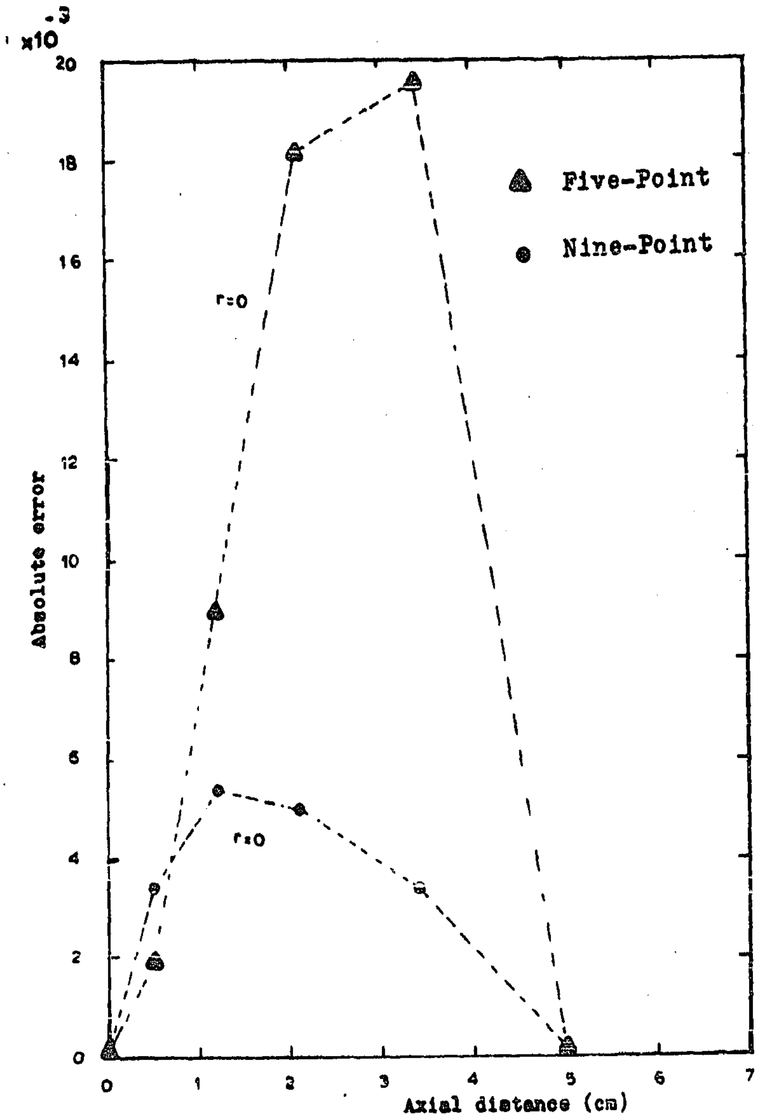


Fig. 41. Absolute error as a function of axial distance for r = 0 and unequal spacing cylindrical bare reactor core

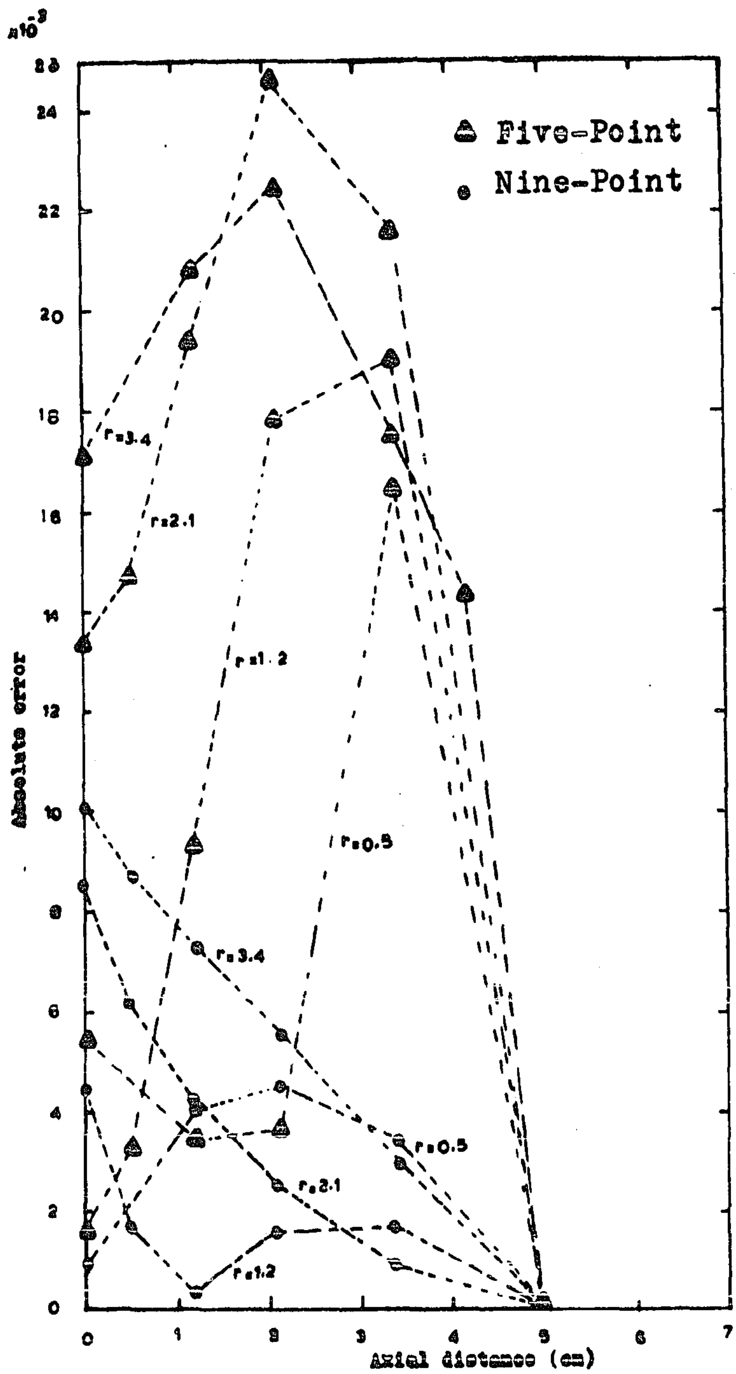


Fig. 42. Absolute error as a function of axial distance for r = 0.5, 1.2, 2.1, 3.4 and unequal spacing cylindrical bare reactor core

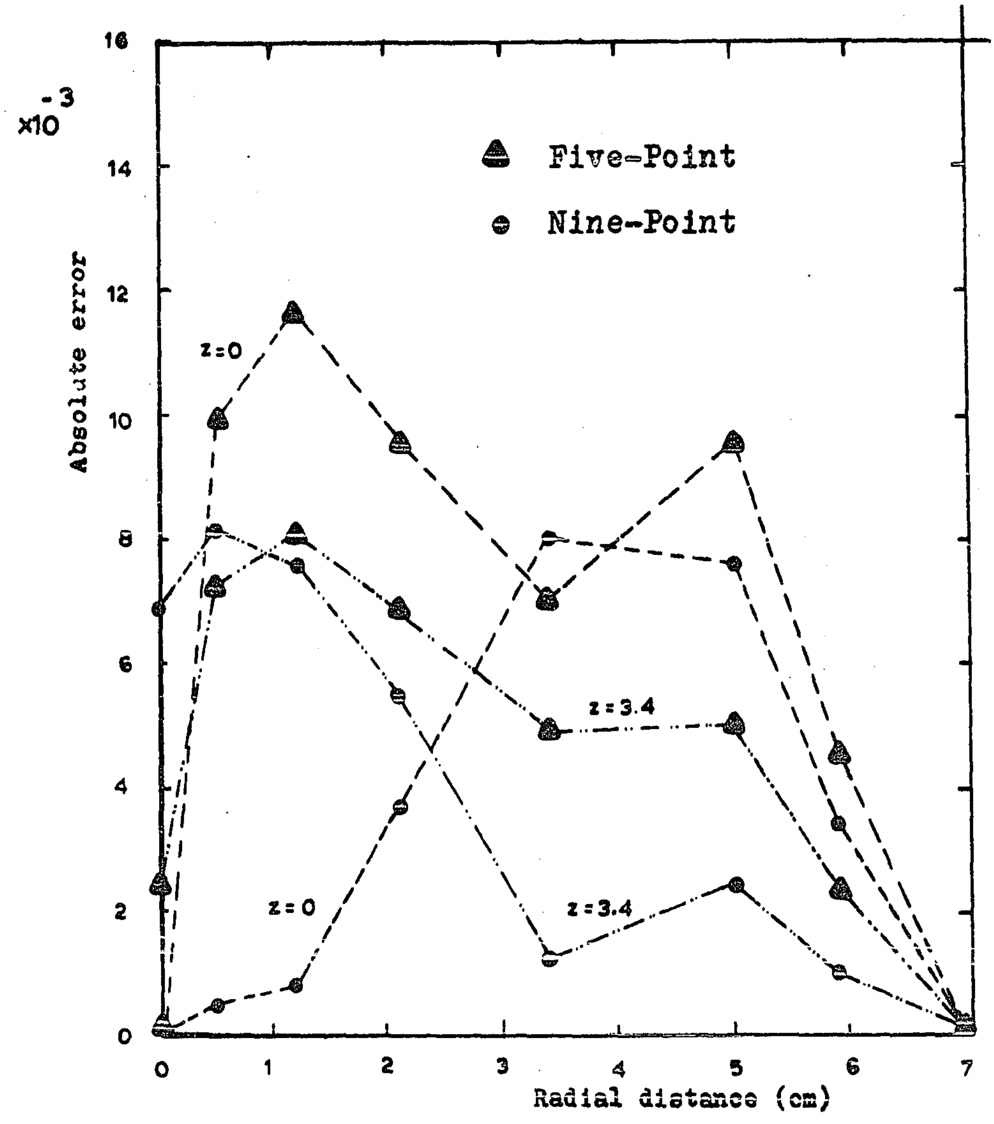


Fig. 43. Absolute error as a function of radial distance for z = 0, 3.4 and unequal spacing in a radially reflected cylindrical reactor core

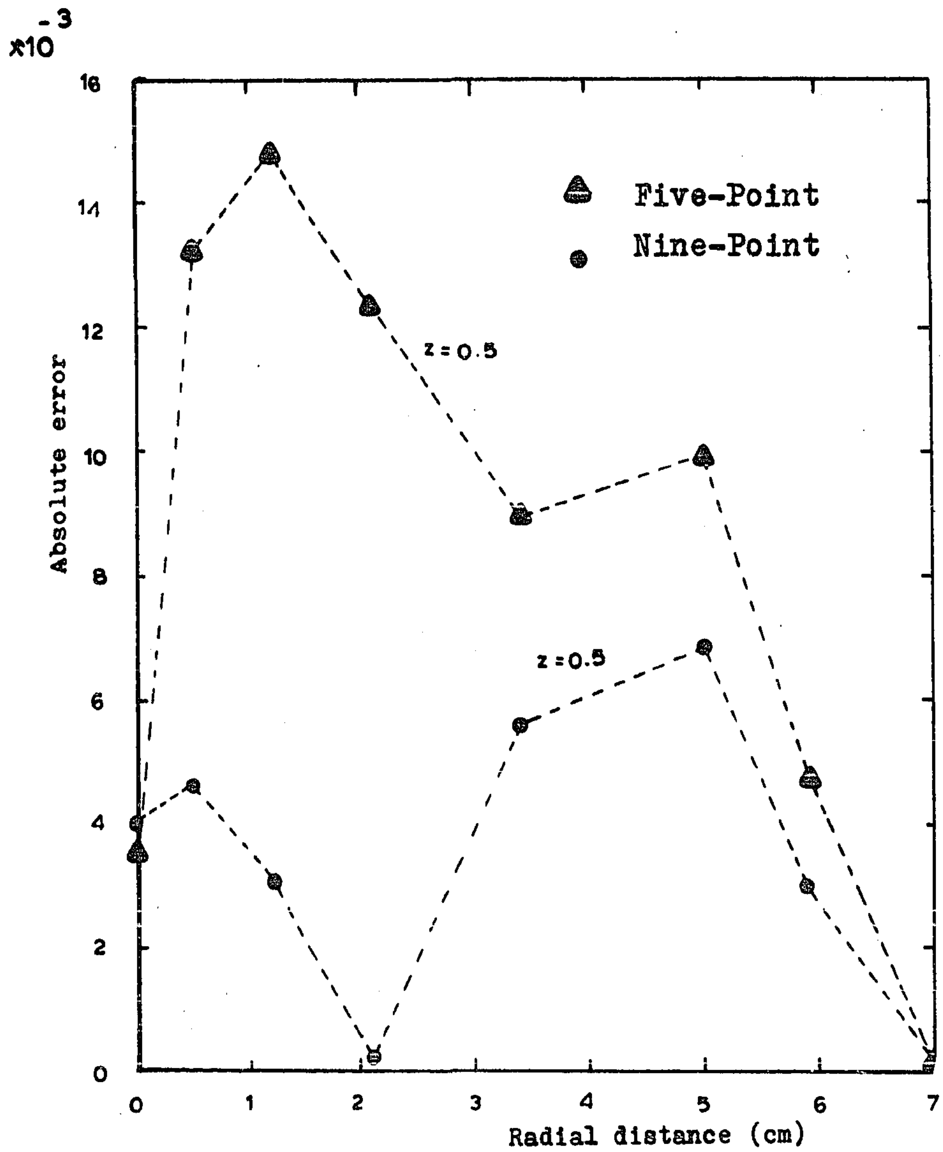


Fig. 44. Absolute error as a function radial distance for $z = 0.5$ and unequal spacing radially reflected cylindrical reactor core

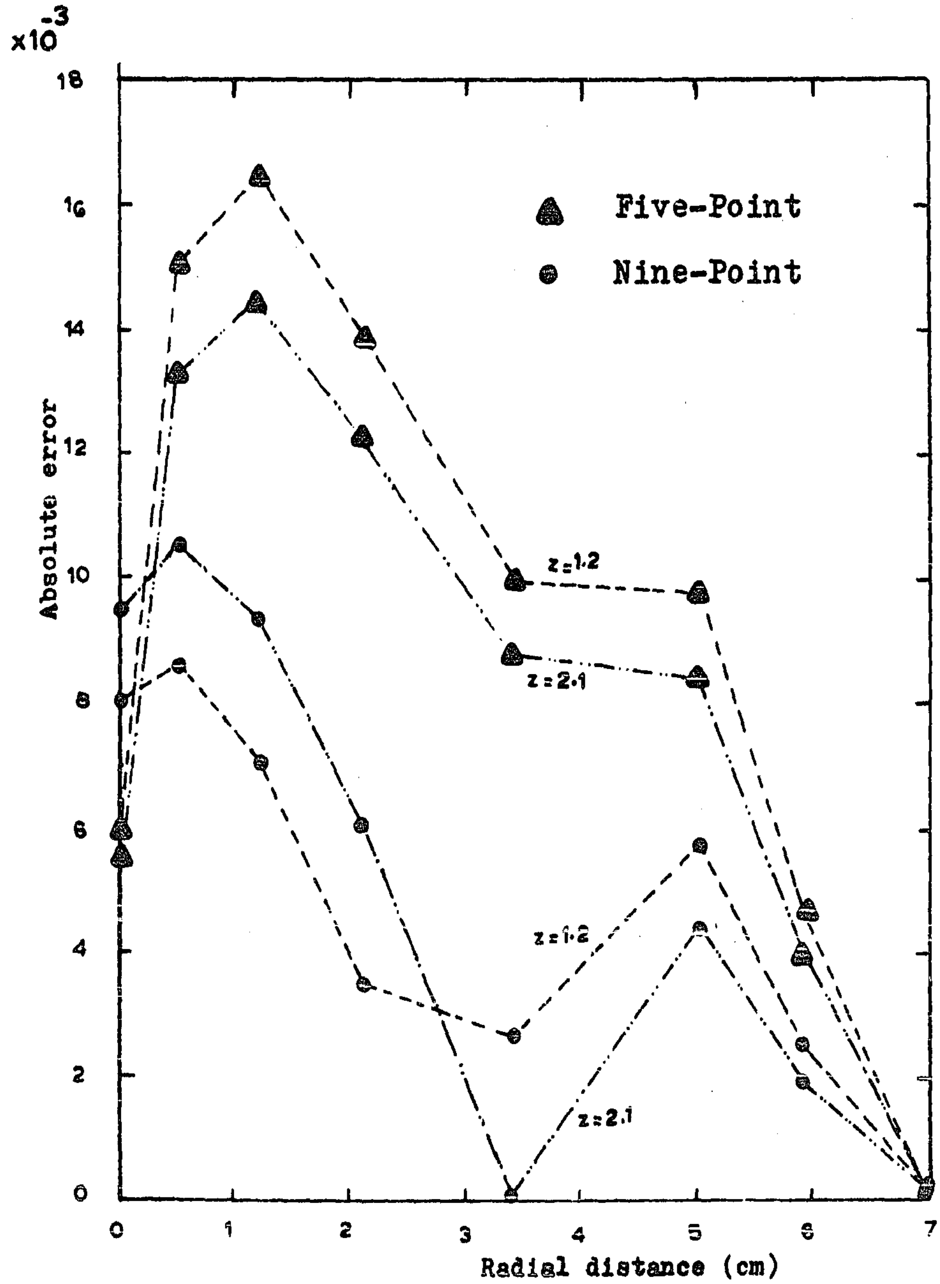


Fig. 45. Absolute error as a function of radial distance for z = 1.2, 2.1 and unequal spacing radially reflected cylindrical reactor core

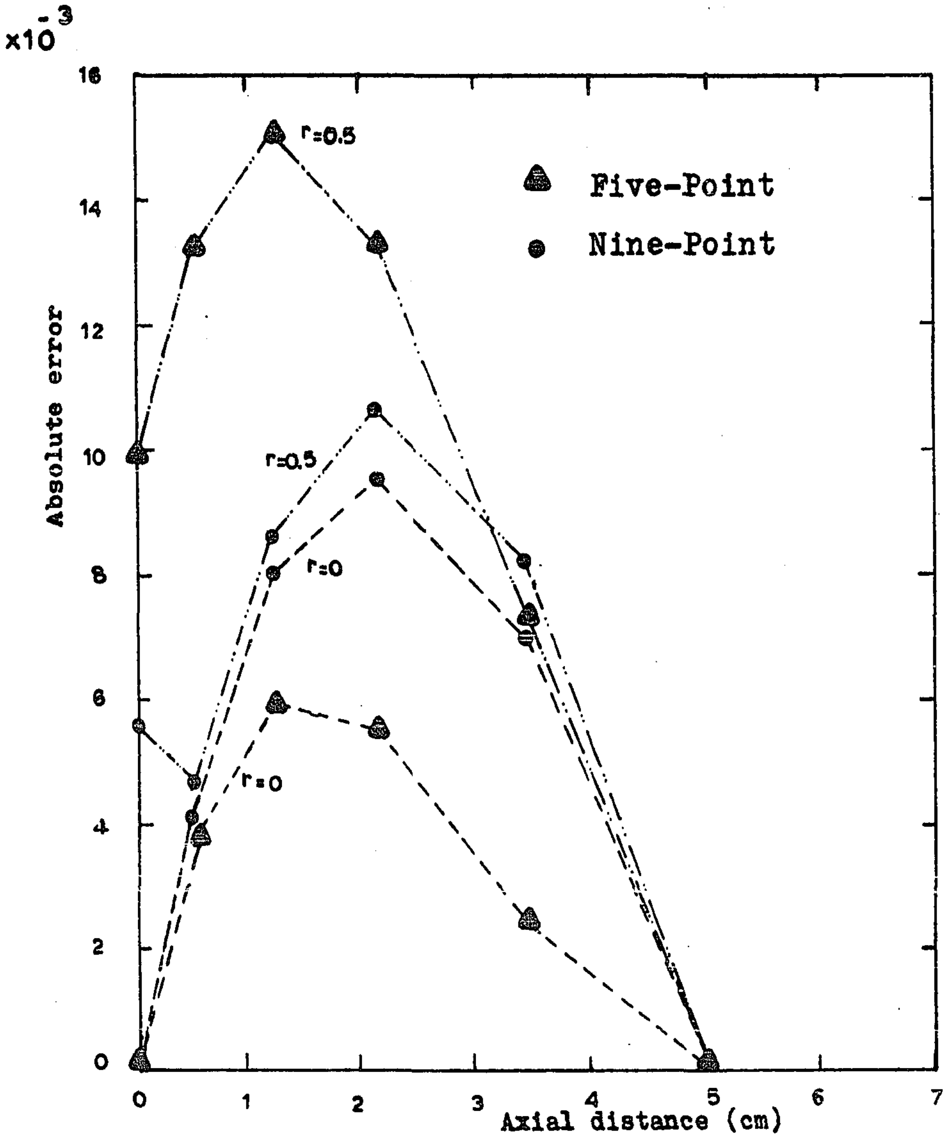


Fig. 46. Absolute error as a function of axial distance for r = 0, 0.5 and unequal spacing radially reflected cylindrical reactor core

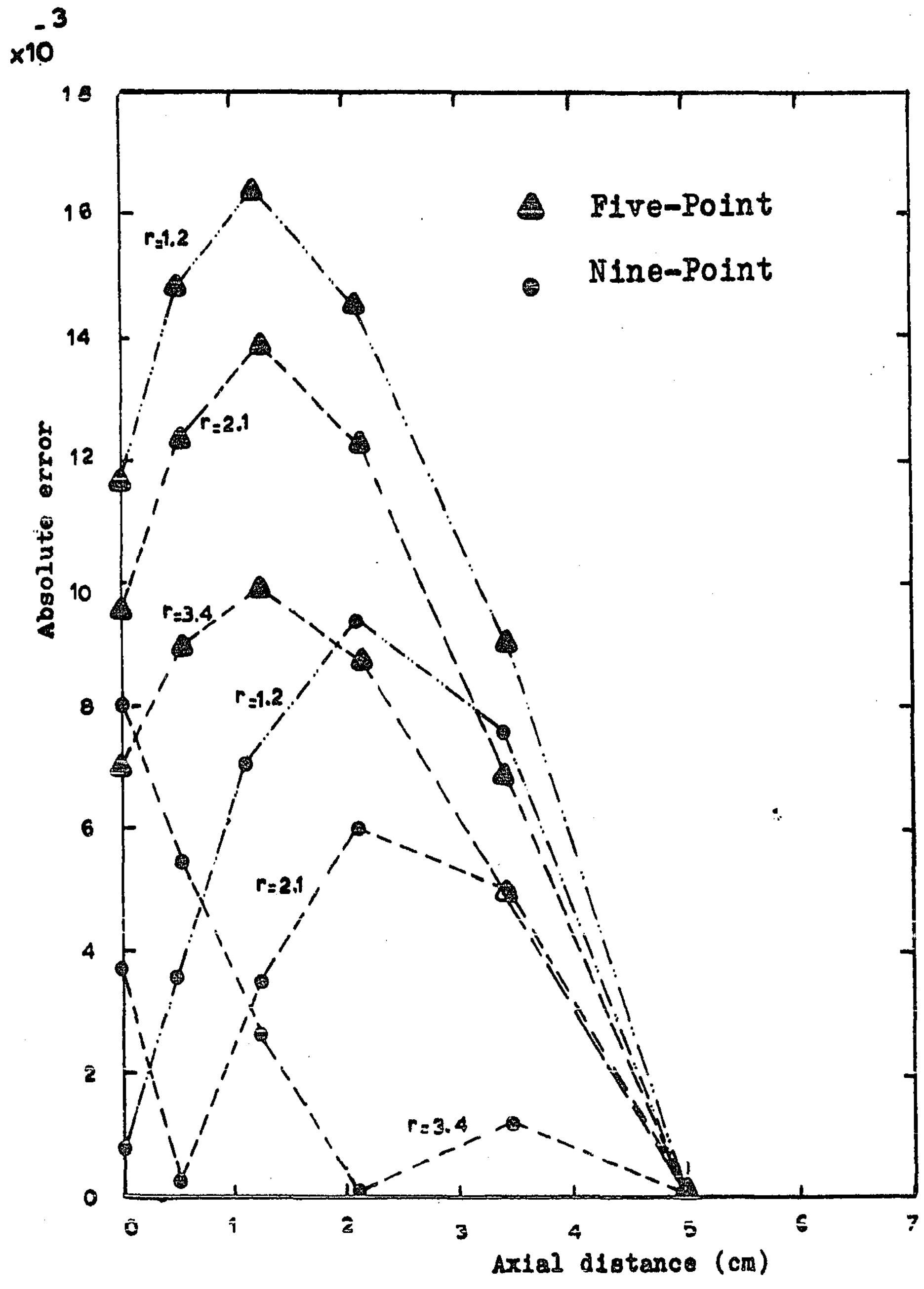


Fig. 47. Absolute error as a function of axial distance for r = 1.2, 2.1, 3.4 and unequal spacing radially reflected cylindrical reactor core

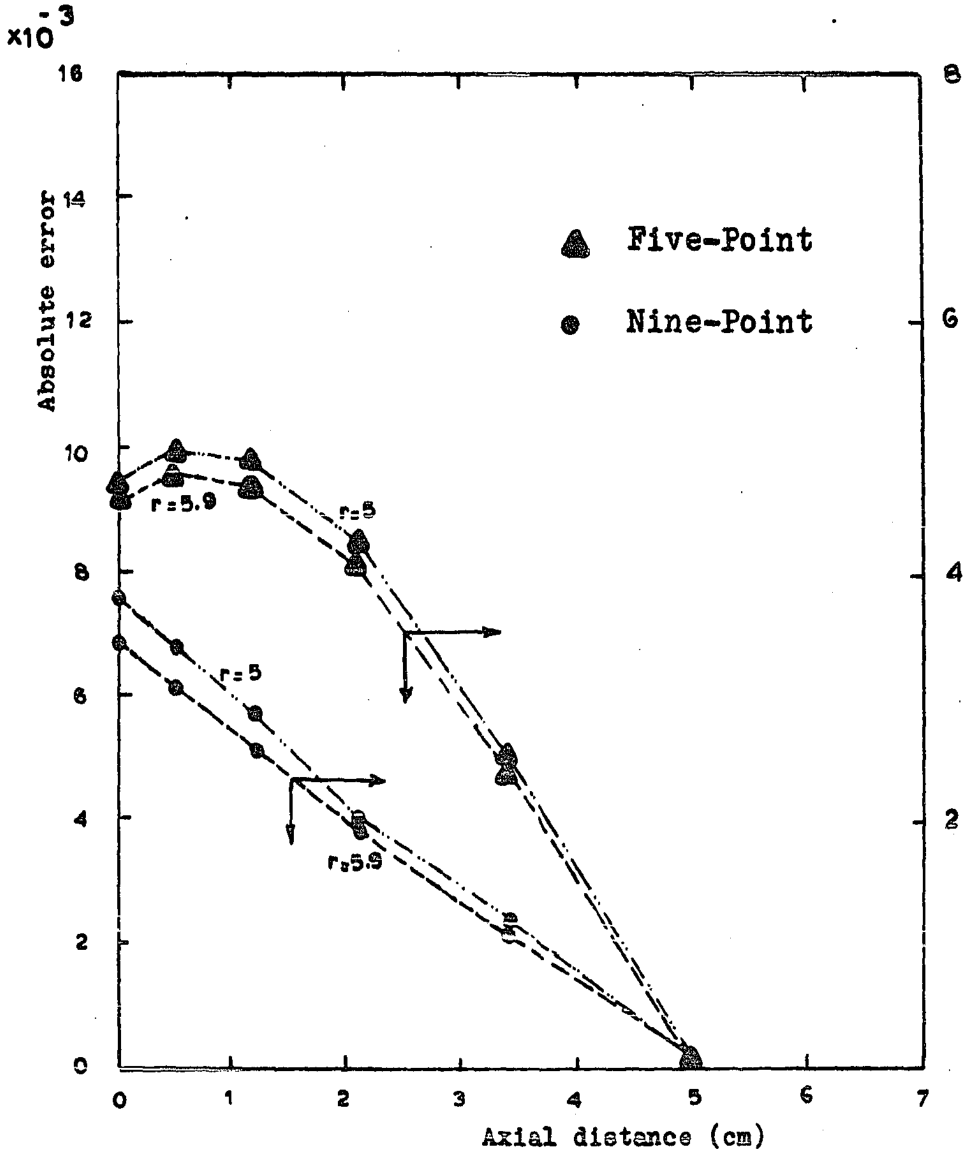


Fig. 48. Absolute error as a function of axial distance for r = 5, 5.9 and unequal spacing radially reflected cylindrical reactor core

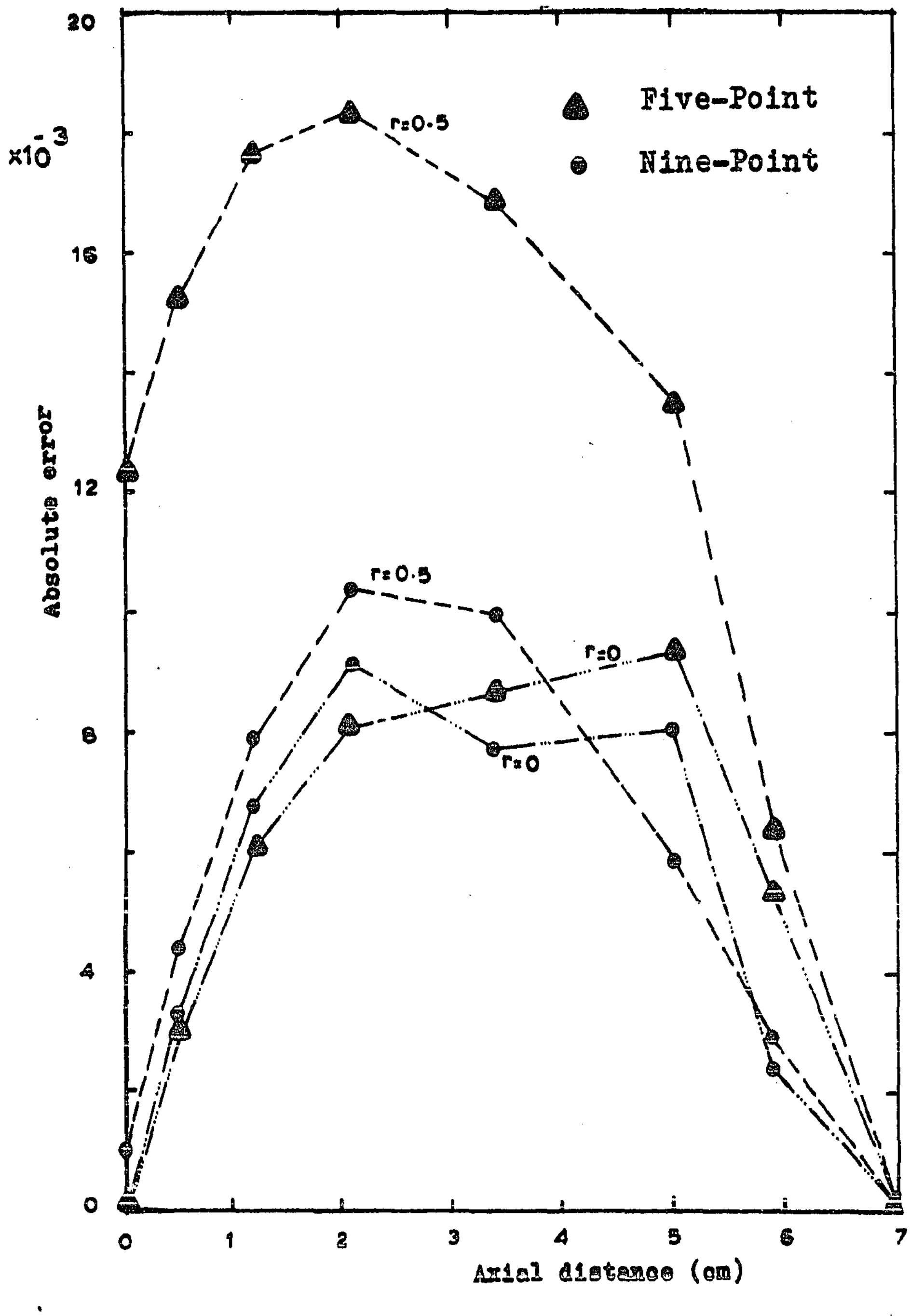


Fig. 49. Absolute error as a function of axial distance for r = 0, 0.5 and unequal spacing in an axially reflected cylindrical reactor core

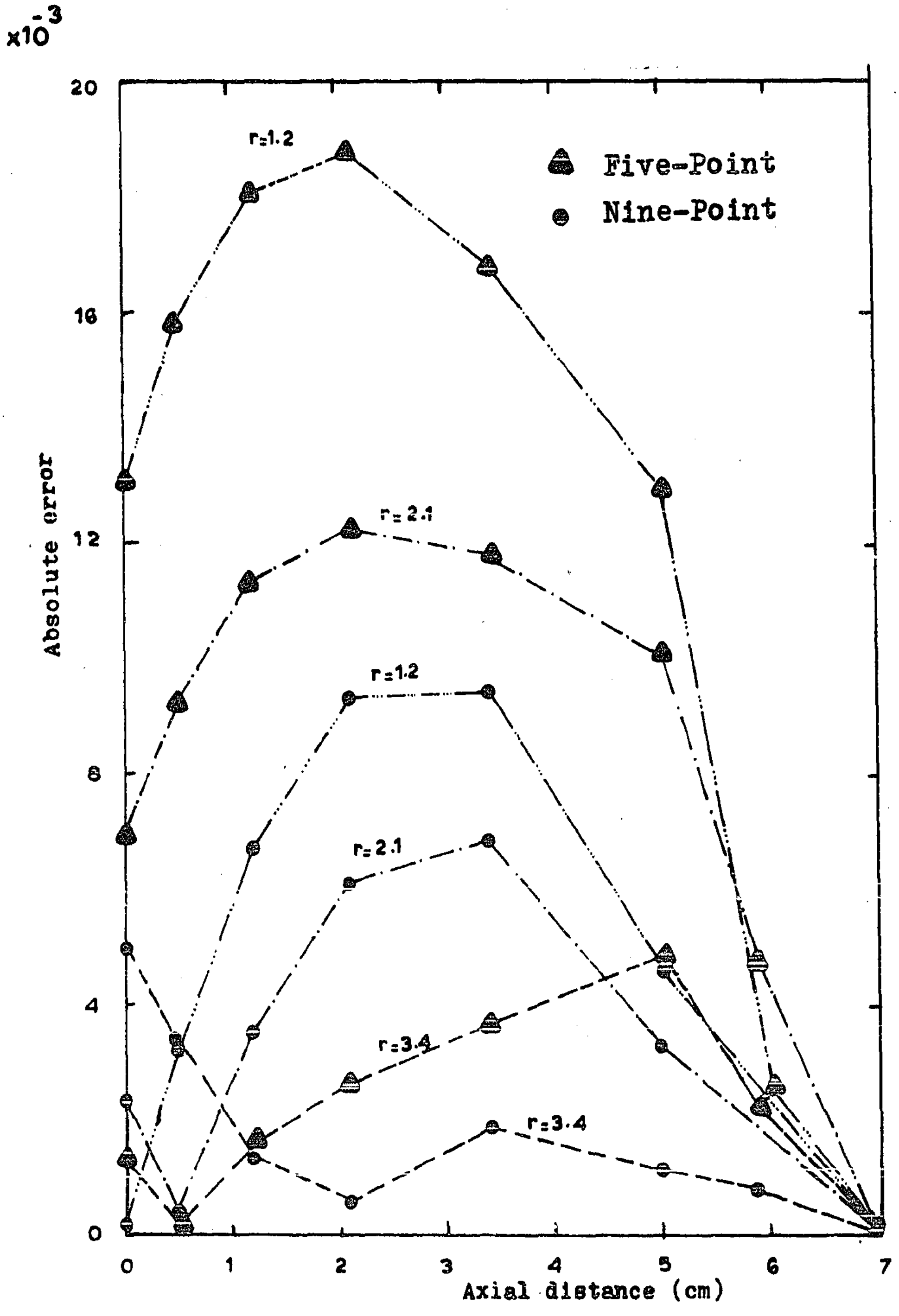


Fig. 50. Absolute error as a function of axial distance for r = 1.2, 2.1, 3.4 and unequal spacing axially reflected cylindrical reactor core

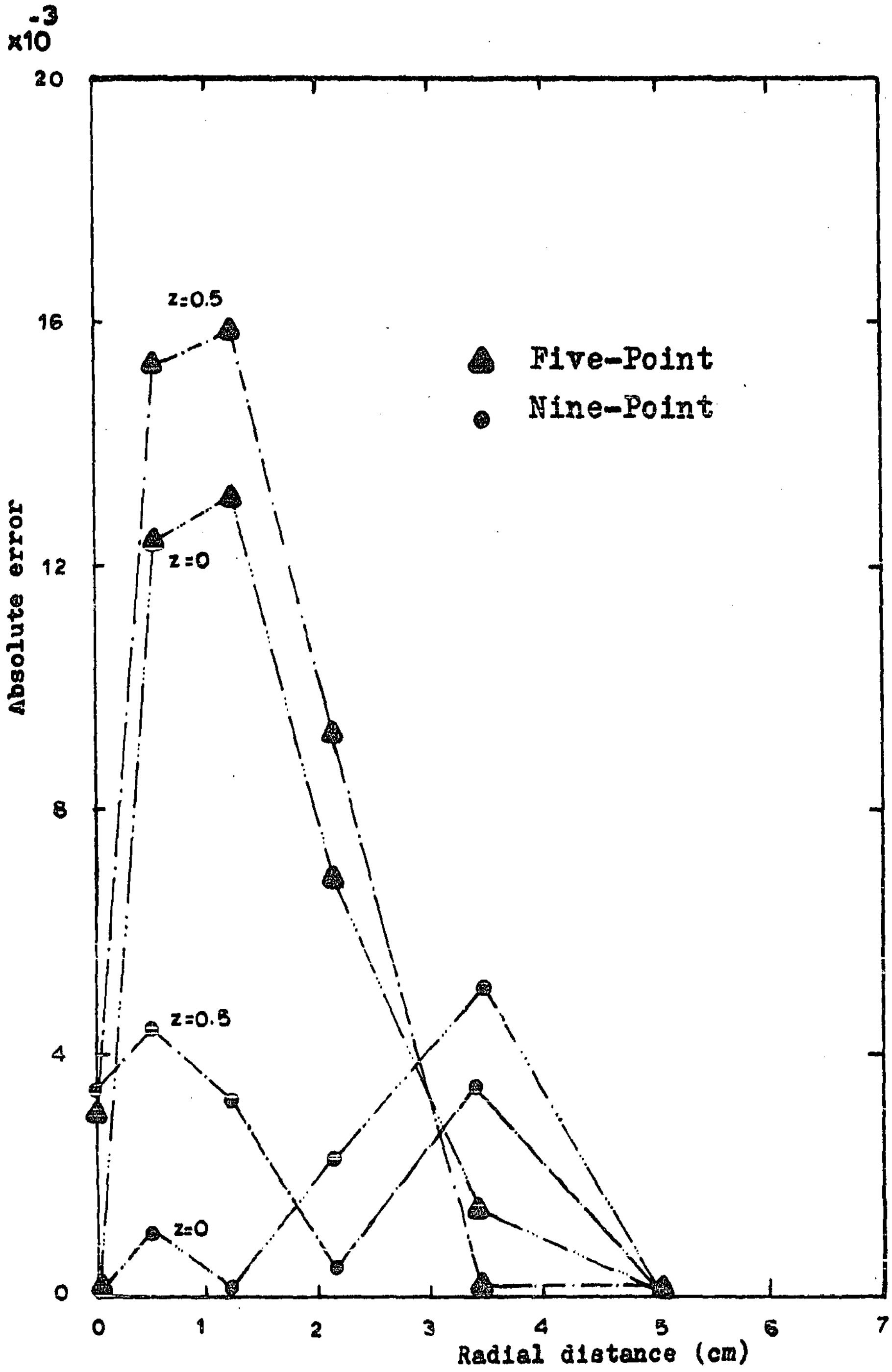


Fig. 51. Absolute error as a function of radial distance for z = 0, 0.5 and unequal spacing axially reflected cylindrical reactor core

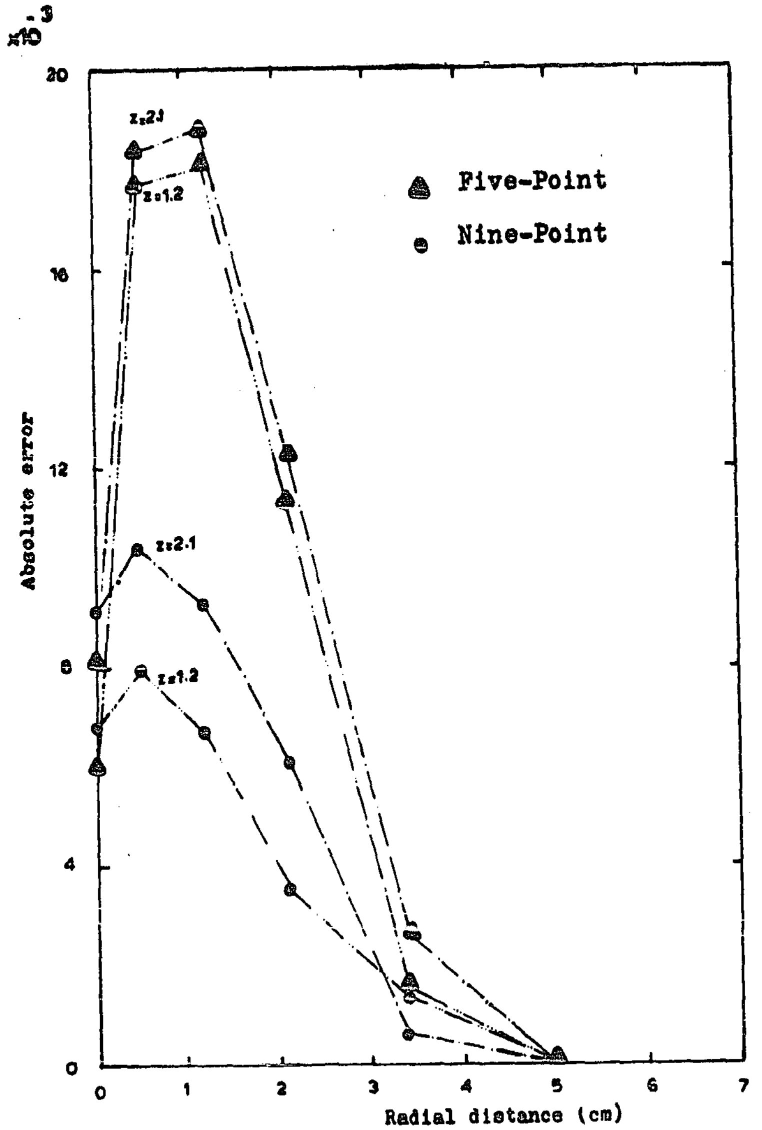


Fig. 52. Absolute error as a function of radial distance for r = 1.2, 2.1 and unequal spacing axially reflected cylindrical reactor core

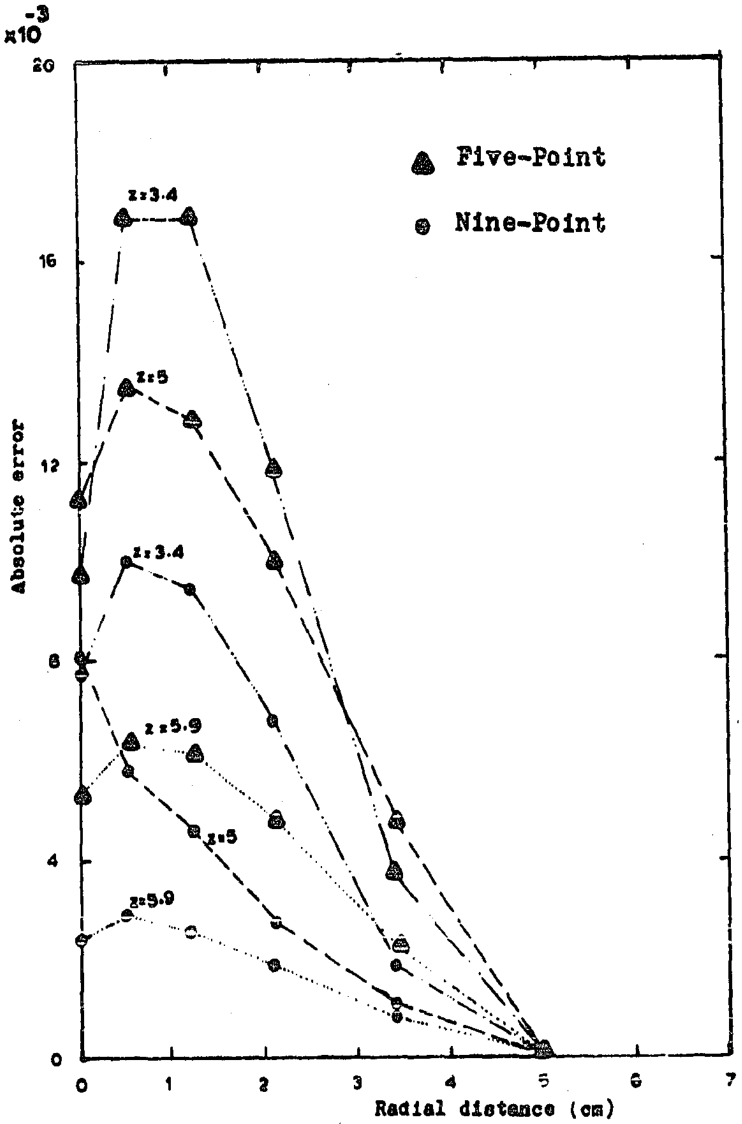


Fig. 53. Absolute error as a function of radial distance for r = 3.4, 5, 5.9 and unequal spacing axially reflected cylindrical reactor core

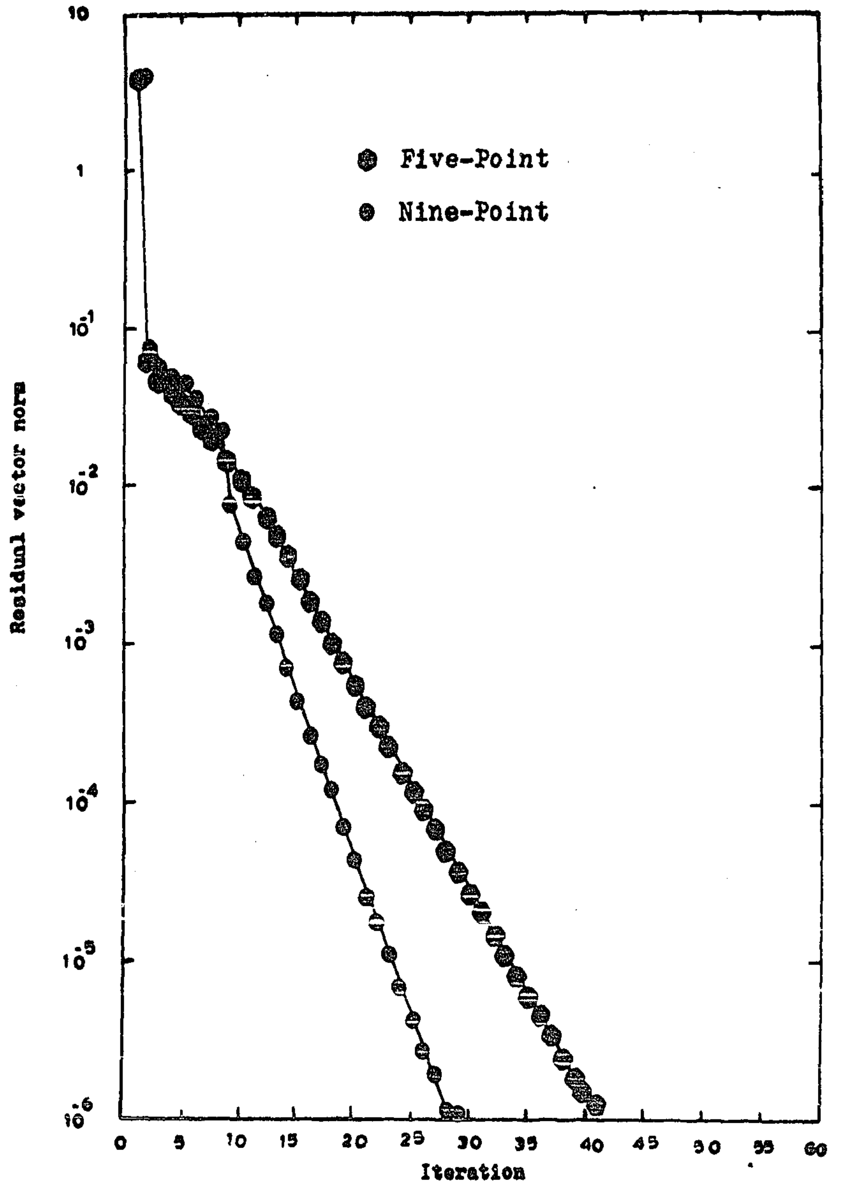


Fig. 54. Residual vector norm as a function of iteration and unequal spacings cylindrical bare reactor core

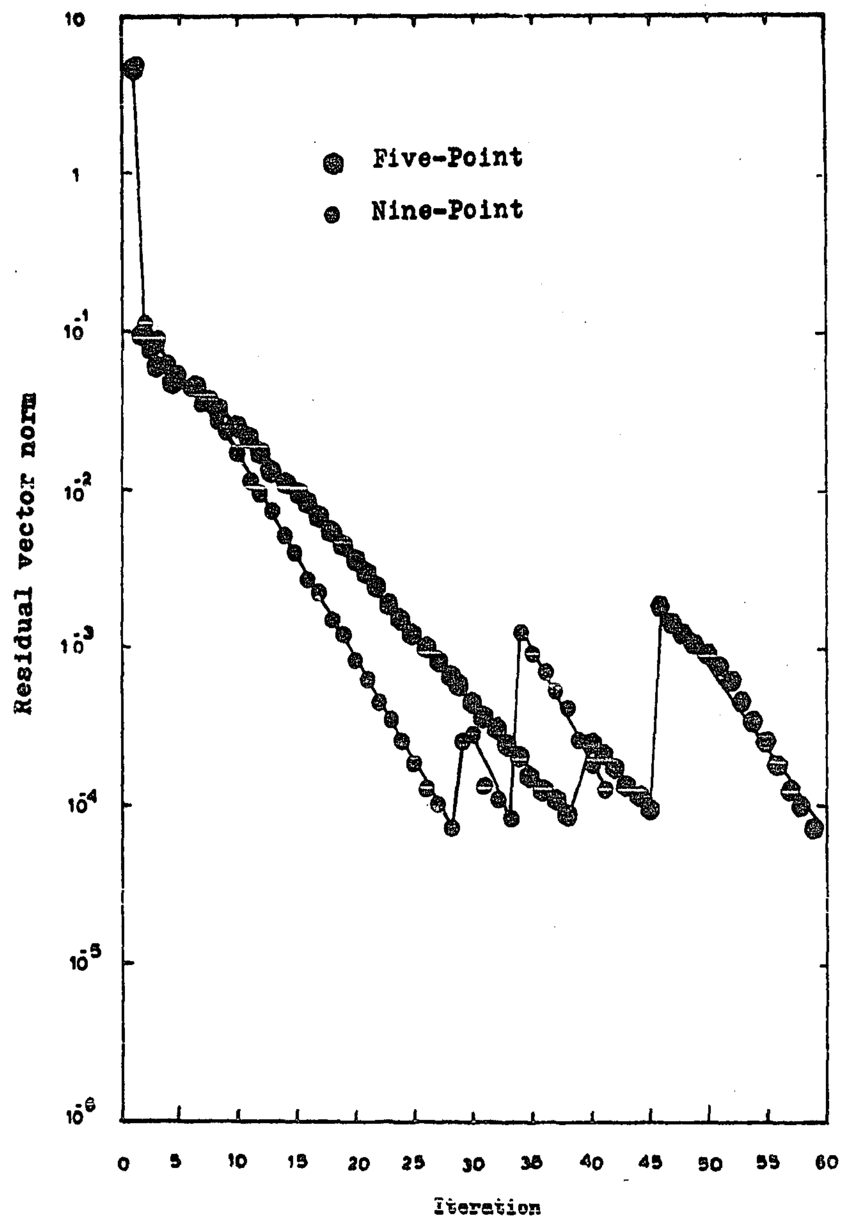


Fig. 55. Residual vector norm as a function of iteration and unequal spacing radially reflected cylindrical reactor core

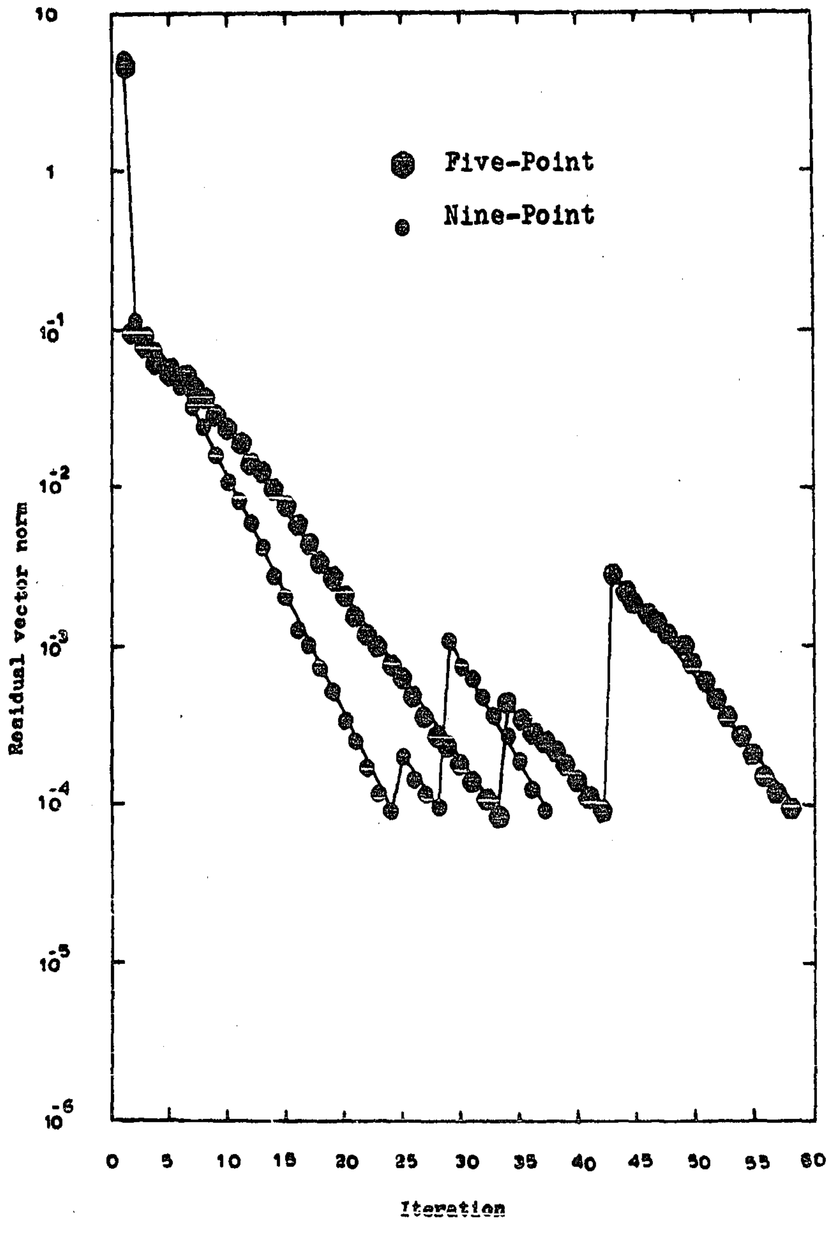


Fig. 56. Residual vector norm as a function of iteration and unequal spacing axially reflected cylindrical reactor core

## A. Completely-Reflected Cylindrical Core

In this case, the completely-reflected cylindrical core is considered as shown in Fig. 57. Equal and unequal spacing along the radial and axial axes are considered as shown in Figs. 58-59, and 66-67. The results for this case may be arranged as follows:

### 1. Equal spacing

The neutron flux values for this case are calculated using the nine-point formula and the five-point formula and tabulated in Tables 19-20. Graphs for the neutron flux versus radial (z = 0) and axial (r = 0) distances are shown in Figs. 60-63. Also, graphs for the residual vector norm versus iteration are shown in Figs. 64-65.

The purpose of this calculation is to show the validity of the nine-point formula for a completely reflected cylindrical core. The analytical solution for this kind of calculation can not be determined. No comparison has been made between the nine-point formula and the five-point formula with respect to accuracy.

The convergence rates as calculated from the slope of the curve of Figs. 64-65 for the nine-point formula are 0.459 and 0.397 and for the five-point formula are 0.299 and 0.287, respectively. These results indicate that the nine-point formula converges faster than the five-point formula.

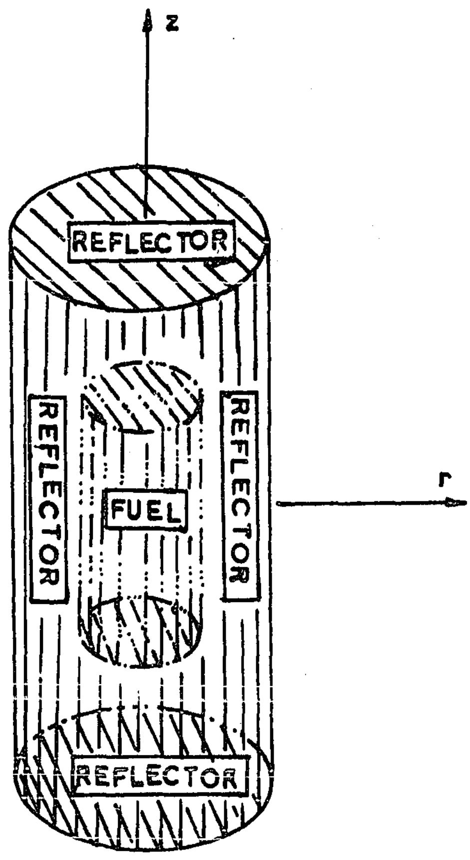


Fig. 57. Completely reflected cylindrical reactor core

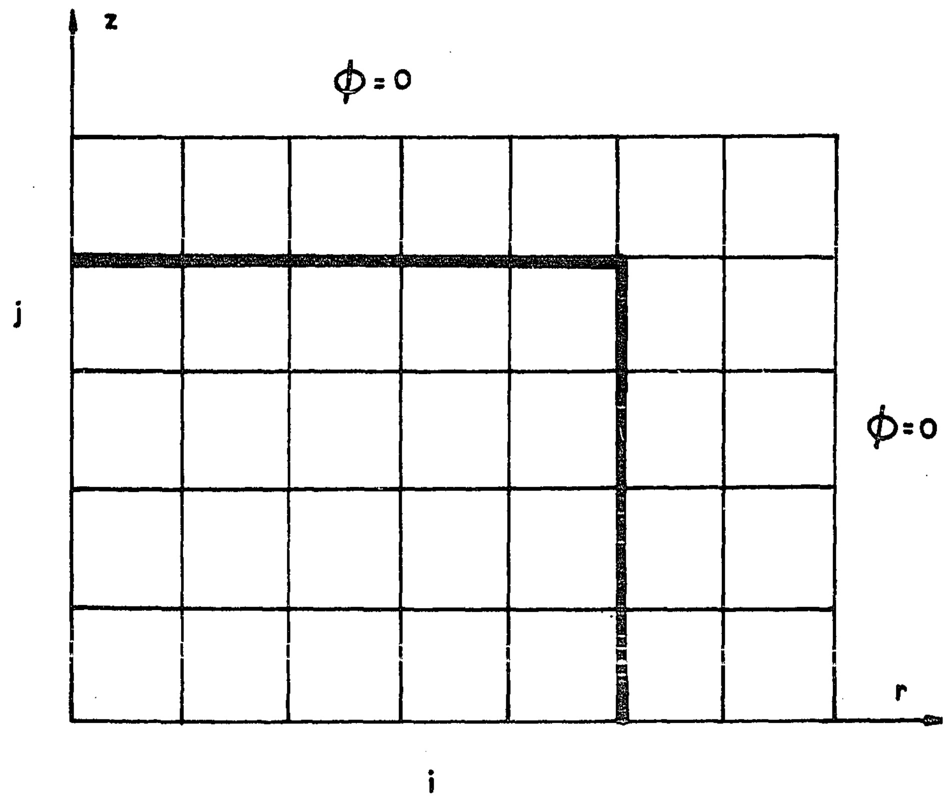


Fig. 58. Equal spacing with more spacing along the radial axis for a completely reflected cylindrical reactor core

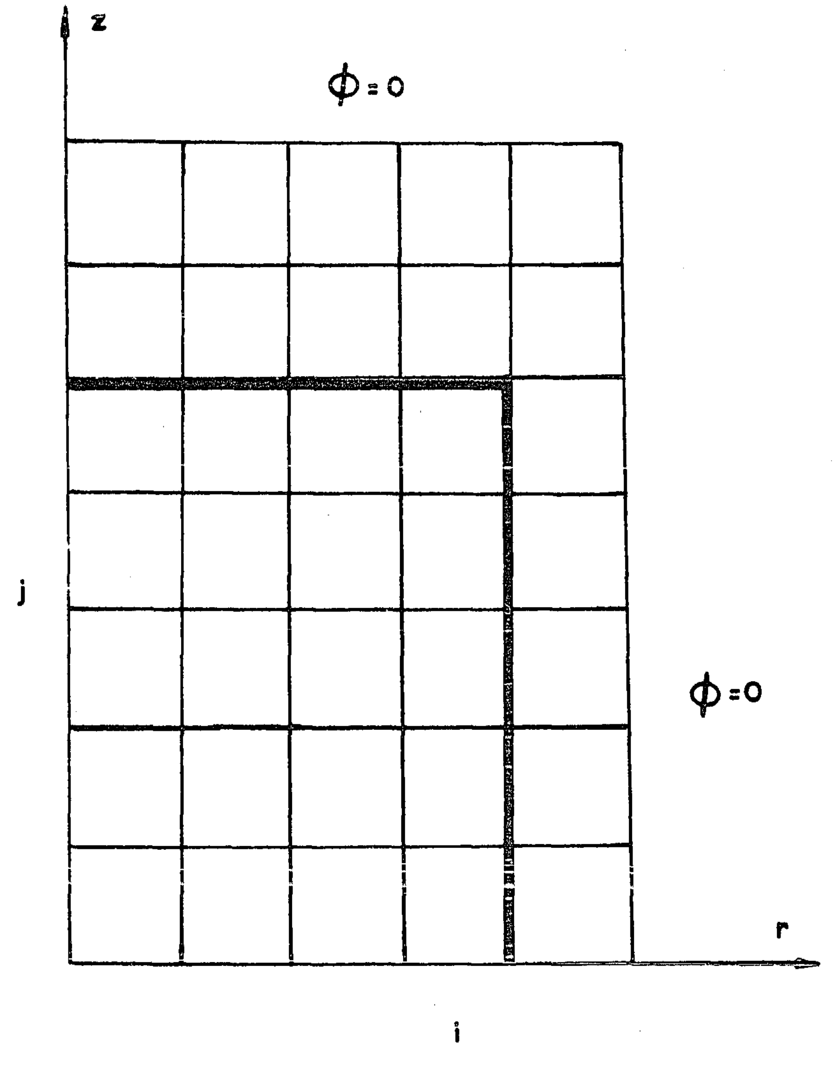


Fig. 59. Equal spacing with more spacings along the axial axis for a completely reflected cylindrical reactor core

Table 19. The neutron flux distribution in a completely reflected cylindrical core for equal spacing along the radial and axial axes and input data of $D_c$ = 0.5 cm, $\Sigma_c$ = 0.1385 $cm^{-1}$, $D_e$ = 1.0 cm, $\Sigma_e$ = 0.1 $cm^{-1}$

| Method | z \ r | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| NPS | 0 | 1.000 | 0.954 | 0.829 | 0.638 | 0.407 | 0.162 | 0.067 | 0.000 |
| | 1 | 0.939 | 0.896 | 0.778 | 0.599 | 0.382 | 0.153 | 0.063 | 0.000 |
| | 2 | 0.762 | 0.727 | 0.632 | 0.487 | 0.311 | 0.125 | 0.052 | 0.000 |
| | 3 | 0.491 | 0.468 | 0.407 | 0.315 | 0.203 | 0.083 | 0.035 | 0.000 |
| | 4 | 0.155 | 0.152 | 0.133 | 0.103 | 0.067 | 0.036 | 0.018 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.915 | 0.782 | 0.595 | 0.373 | 0.144 | 0.059 | 0.000 |
| | 1 | 0.938 | 0.858 | 0.733 | 0.558 | 0.350 | 0.135 | 0.056 | 0.000 |
| | 2 | 0.759 | 0.695 | 0.594 | 0.452 | 0.284 | 0.111 | 0.046 | 0.000 |
| | 3 | 0.486 | 0.445 | 0.380 | 0.290 | 0.184 | 0.074 | 0.031 | 0.000 |
| | 4 | 0.152 | 0.1387 | 0.119 | 0.091 | 0.060 | 0.032 | 0.015 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 20. The neutron flux distribution in a completely reflected cylindrical core for equal spacing along the axial and radial axes and input data of $D_c$ = 0.5 cm, $\Sigma_c$ = 0.1536 cm$^{-1}$, $D_e$ = 1.0 cm, $\Sigma_e$ = 0.1 cm$^{-1}$

| Method | r \ z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| NPS | 0 | 1.000 | 0.962 | 0.851 | 0.676 | 0.448 | 0.181 | 0.077 | 0.000 |
| | 1 | 0.926 | 0.891 | 0.788 | 0.626 | 0.415 | 0.172 | 0.071 | 0.000 |
| | 2 | 0.726 | 0.699 | 0.619 | 0.491 | 0.327 | 0.136 | 0.057 | 0.000 |
| | 3 | 0.441 | 0.425 | 0.376 | 0.299 | 0.201 | 0.086 | 0.037 | 0.000 |
| | 4 | 0.129 | 0.124 | 0.110 | 0.088 | 0.060 | 0.034 | 0.017 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.962 | 0.850 | 0.673 | 0.444 | 0.179 | 0.076 | 0.000 |
| | 1 | 0.863 | 0.830 | 0.733 | 0.581 | 0.383 | 0.155 | 0.066 | 0.000 |
| | 2 | 0.658 | 0.633 | 0.560 | 0.444 | 0.294 | 0.120 | 0.051 | 0.000 |
| | 3 | 0.391 | 0.376 | 0.333 | 0.264 | 0.176 | 0.074 | 0.032 | 0.000 |
| | 4 | 0.108 | 0.104 | 0.092 | 0.073 | 0.051 | 0.028 | 0.014 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

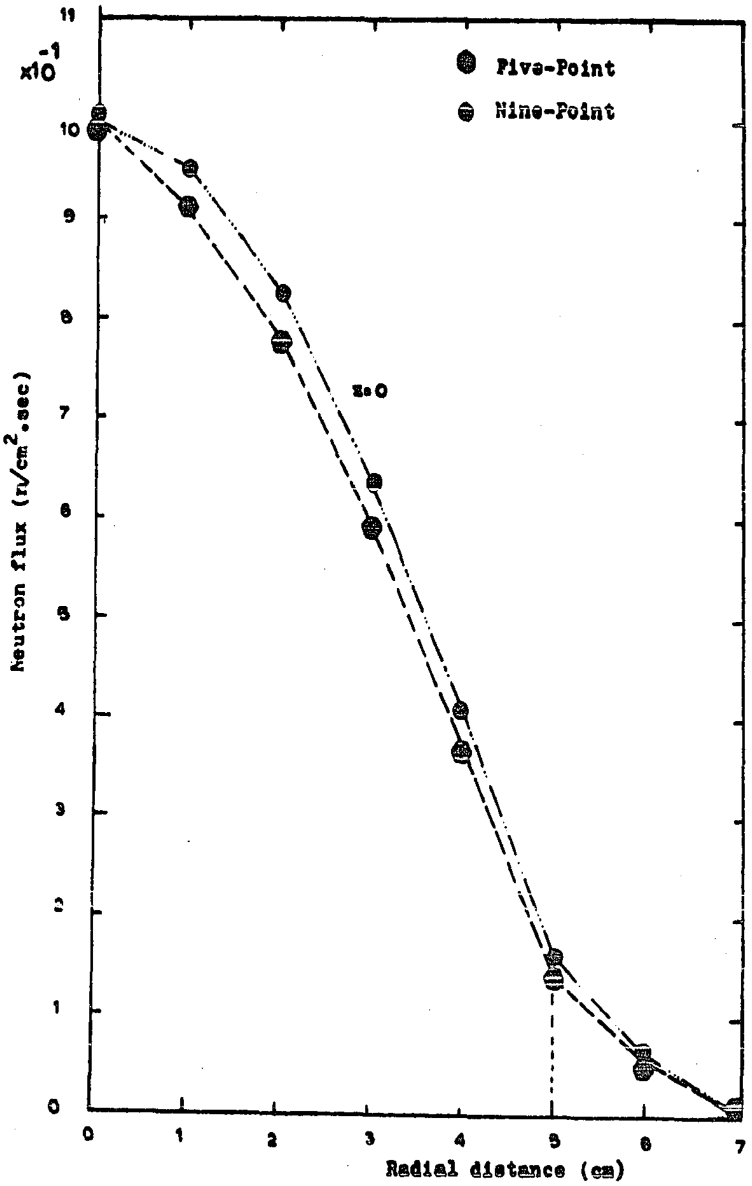


Fig. 60. Neutron flux as a function of radial distance for $z = 0$ and equal spacing completely reflected cylindrical reactor core with more spacings along the radial axis

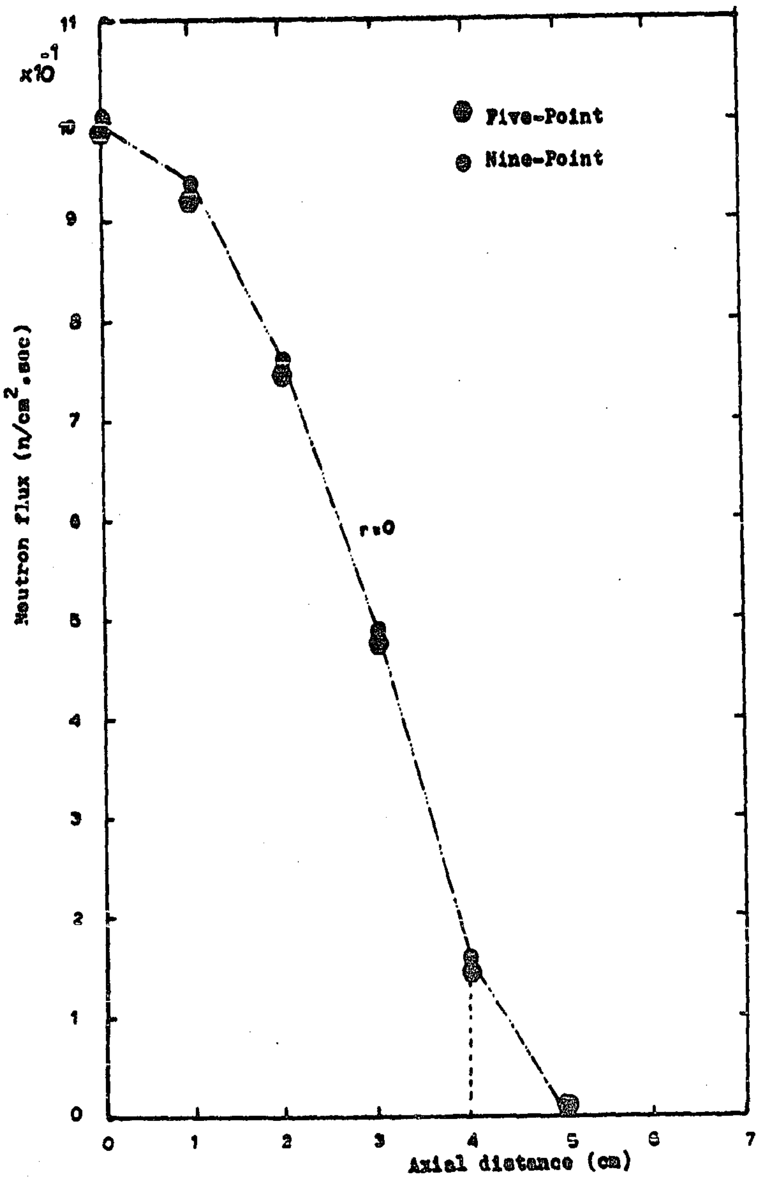


Fig. 61. Neutron flux as a function of axial distance for $r = 0$ and equal spacing in a completely reflected cylindrical reactor core

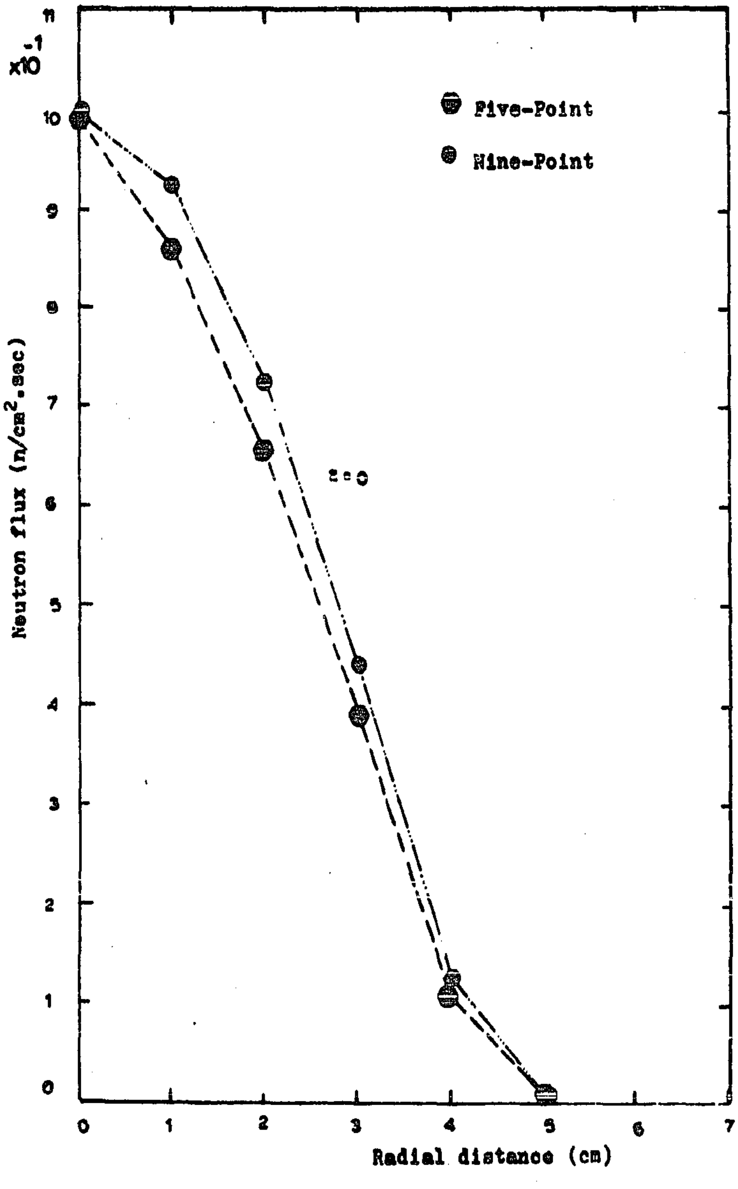


Fig. 62. Neutron flux as a function of radial distance for z = 0 and equal spacing in a completely reflected cylindrical reactor core

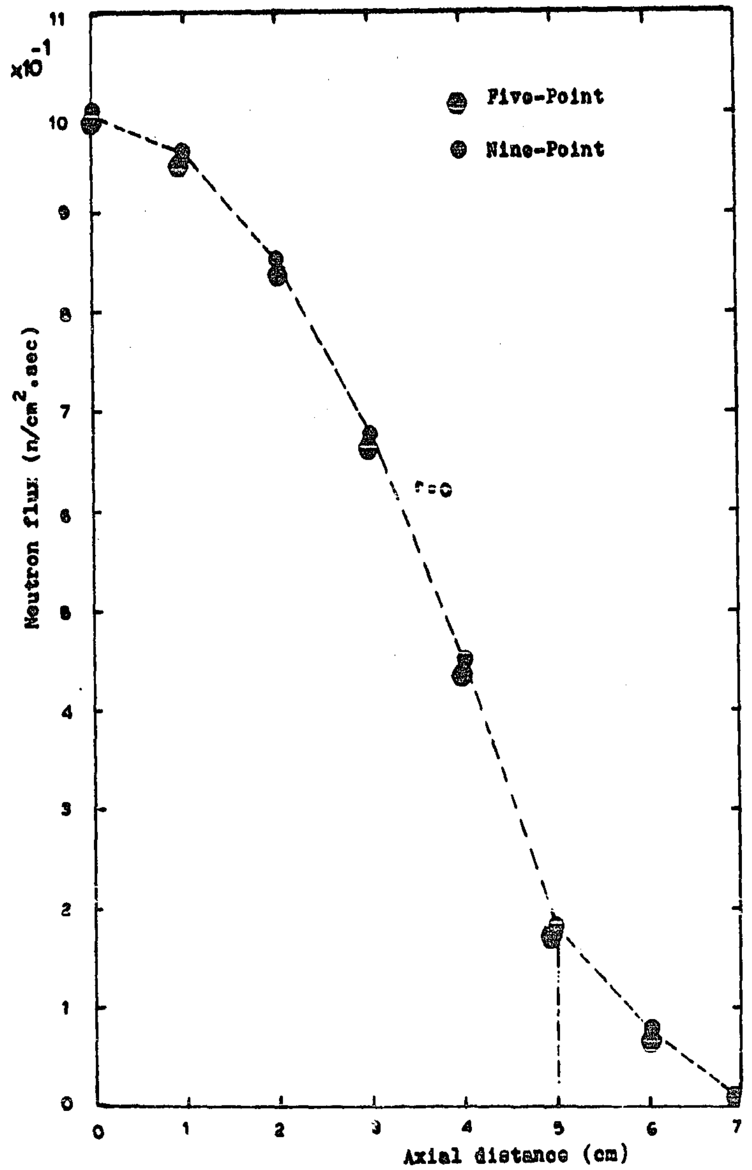


Fig. 63. Neutron flux as a function of axial distance for r = 0 and equal spacing in a completely reflected cylindrical reactor core with more spacings along the axial axis

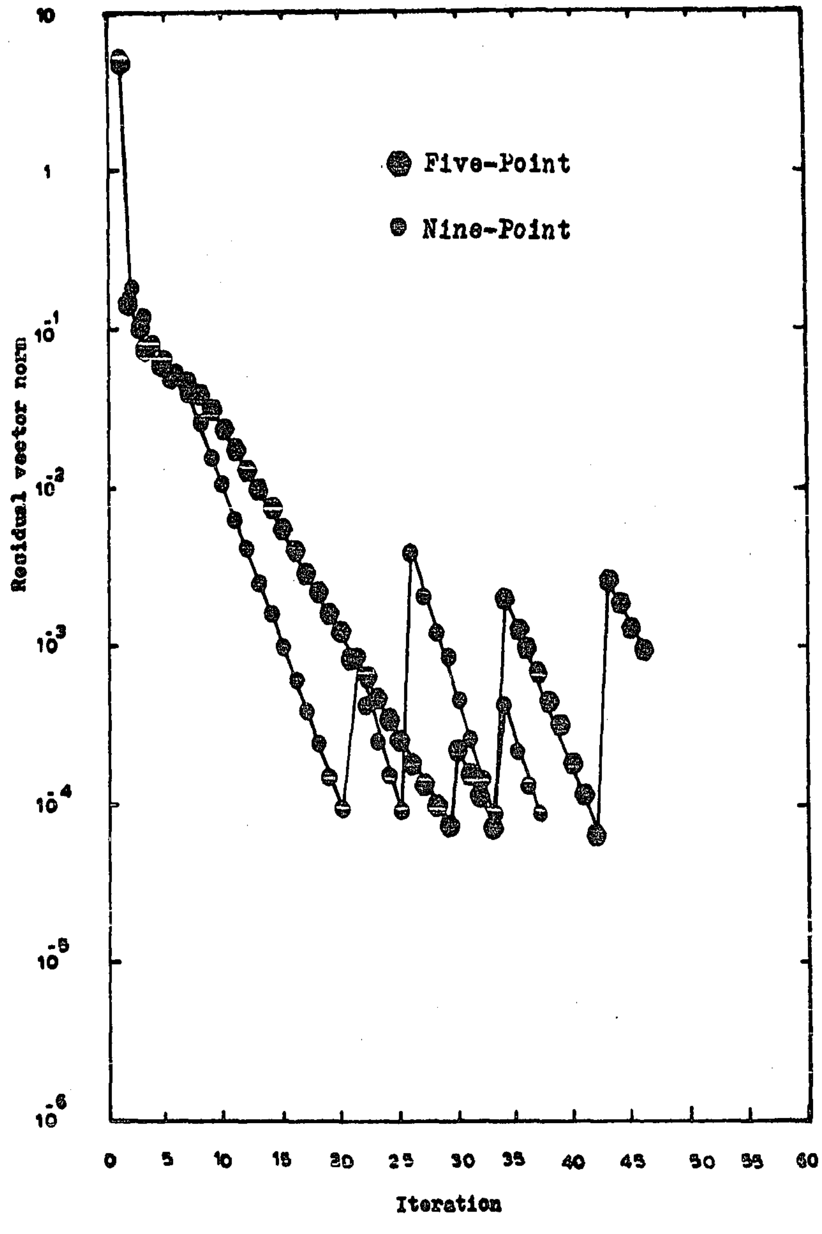


Fig. 64. Residual vector norm as a function of iteration and equal spacing in a completely reflected cylindrical reactor core with more spacings along the radial axis

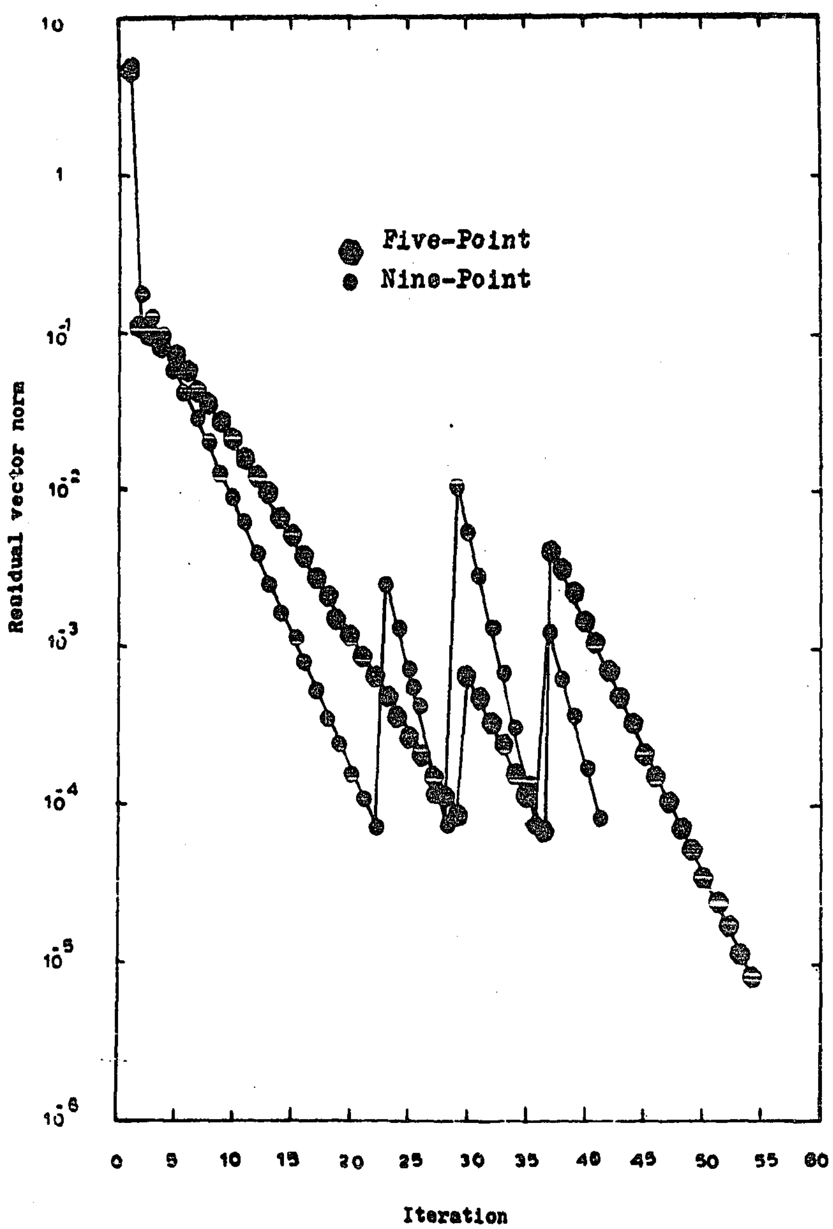


Fig. 65. Residual vector norm as a function of iteration and equal spacing in a completely reflected cylindrical reactor core with more spacings along the axial axis

## 2. Unequal spacing

The neutron flux values for this case were calculated using the nine-point formula and the five-point formula and tabulated in Tables 21-22. Graphs for the neutron flux versus radial (z = 0) and axial (r = 0) distances are shown in Figs. 68-71. Plots of residual vector norm versus iteration are shown in Figs. 98-99.

This calculation was performed in order to show the validity of the nine-point formula for the completely reflected cylindrical core calculation where the analytical solution can not be determined. No comparison was made between the nine-point formula and five-point formula with respect to accuracy.

The convergence rates as calculated from the slopes of the curves of Figs. 72-73. For the nine-point formula the rates are 0.279 and 0.333 and for the five-point formula the rates are 0.191 and 0.234. Therefore, these results indicate that the nine-point formula converges faster than the five-point formula.

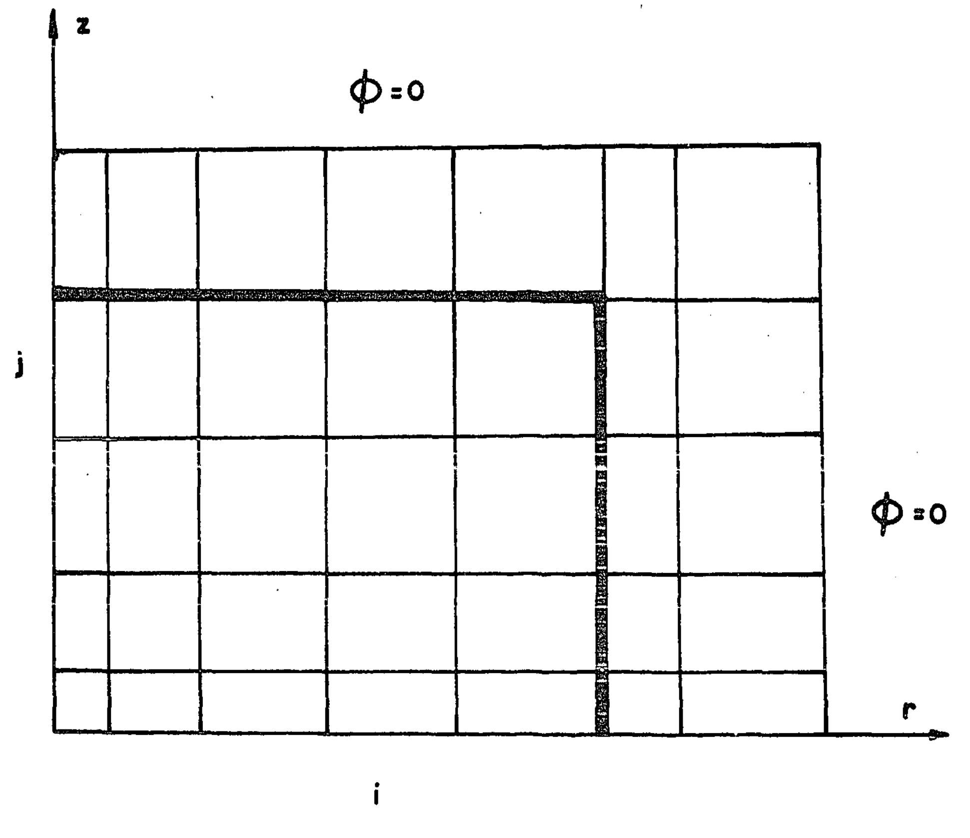


Fig. 66. Unequal spacing mesh with more spacing along the radial axis for a completely reflected cylindrical reactor core

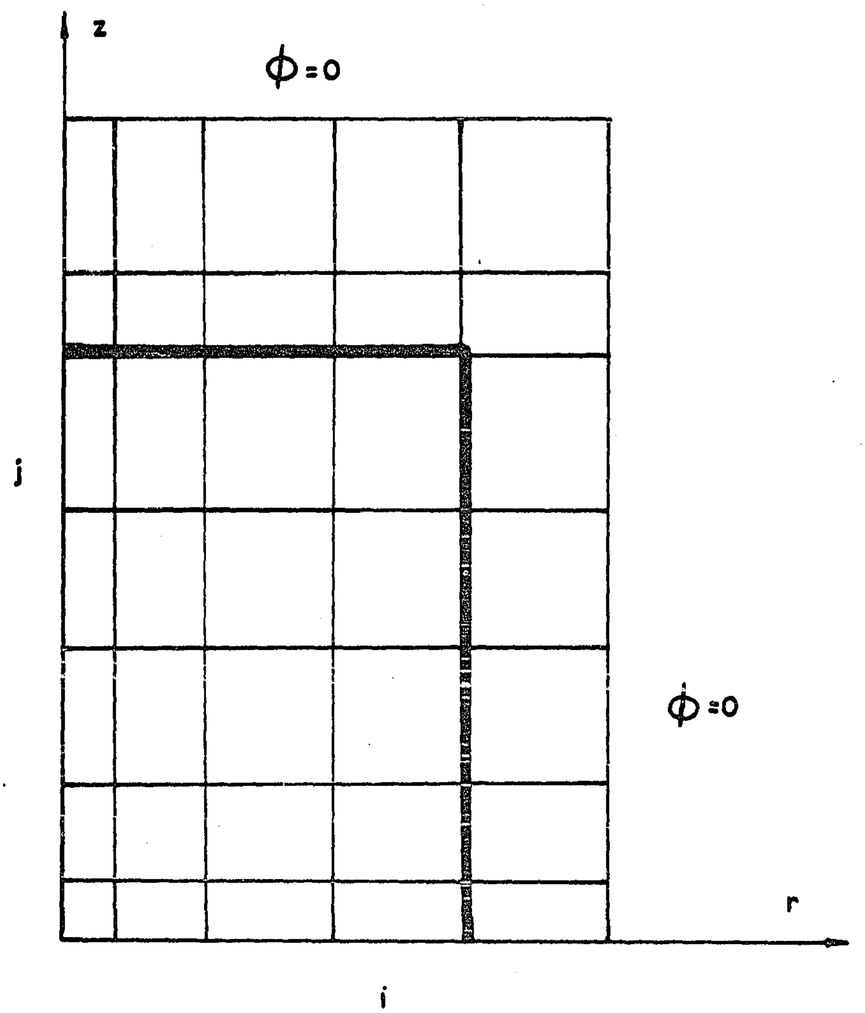


Fig. 67. Unequal spacing mesh with more spacing along the axial axis for a completely reflected cylindrical reactor core

Table 21. The neutron flux distribution in a completely reflected cylindrical core for unequal spacing along the radial and axial axes and input data of $D_c$ = 0.5 cm, $\Sigma_c$ = 0.1385 $cm^{-1}$, $D_e$ = 1.0 cm, $\Sigma_e$ = 0.1 $cm^{-1}$

| Method | z \ r | 0 | 0.5 | 1.2 | 2.1 | 3.4 | 5 | 5.9 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| NPS | 0 | 1.000 | 0.988 | 0.938 | 0.816 | 0.555 | 0.164 | 0.075 | 0.000 |
| | 0.5 | 0.975 | 0.963 | 0.914 | 0.796 | 0.541 | 0.160 | 0.073 | 0.000 |
| | 1.2 | 0.882 | 0.871 | 0.826 | 0.720 | 0.490 | 0.146 | 0.067 | 0.000 |
| | 2.1 | 0.674 | 0.664 | 0.629 | 0.548 | 0.375 | 0.115 | 0.053 | 0.000 |
| | 3.4 | 0.235 | 0.235 | 0.224 | 0.196 | 0.135 | 0.055 | 0.028 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.979 | 0.924 | 0.802 | 0.539 | 0.147 | 0.067 | 0.000 |
| | 0.5 | 0.975 | 0.954 | 0.901 | 0.782 | 0.525 | 0.143 | 0.066 | 0.000 |
| | 1.2 | 0.880 | 0.861 | 0.813 | 0.706 | 0.475 | 0.131 | 0.060 | 0.000 |
| | 2.1 | 0.666 | 0.652 | 0.616 | 0.535 | 0.361 | 0.103 | 0.048 | 0.000 |
| | 3.4 | 0.218 | 0.214 | 0.202 | 0.177 | 0.122 | 0.048 | 0.024 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 22. The neutron flux distribution in completely reflected cylindrical core for unequal spacing along the axial and radial axes and input data of $D_c$ = 0.5 cm, $\Sigma_c$ = 0.1536 $cm^{-1}$, $D_e$ = 1.0 cm $\Sigma_c$ = 0.1 $cm^{-1}$

| Method | r \ z | 0 | 0.5 | 1.2 | 2.1 | 3.4 | 5 | 5.9 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| NPS | 0 | 1.000 | 0.987 | 0.939 | 0.828 | 0.584 | 0.172 | 0.081 | 0.000 |
| | 0.5 | 0.976 | 0.963 | 0.916 | 0.808 | 0.568 | 0.172 | 0.078 | 0.000 |
| | 1.2 | 0.874 | 0.862 | 0.820 | 0.723 | 0.508 | 0.156 | 0.071 | 0.000 |
| | 2.1 | 0.642 | 0.634 | 0.603 | 0.532 | 0.375 | 0.119 | 0.055 | 0.000 |
| | 3.4 | 0.201 | 0.198 | 0.189 | 0.167 | 0.119 | 0.051 | 0.027 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| FPS | 0 | 1.000 | 0.986 | 0.936 | 0.823 | 0.574 | 0.169 | 0.077 | 0.000 |
| | 0.5 | 0.956 | 0.943 | 0.895 | 0.787 | 0.550 | 0.162 | 0.074 | 0.000 |
| | 1.2 | 0.846 | 0.835 | 0.793 | 0.698 | 0.488 | 0.145 | 0.067 | 0.000 |
| | 2.1 | 0.614 | 0.605 | 0.576 | 0.507 | 0.356 | 0.109 | 0.051 | 0.000 |
| | 3.4 | 0.169 | 0.166 | 0.158 | 0.140 | 0.101 | 0.043 | 0.023 | 0.000 |
| | 5 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

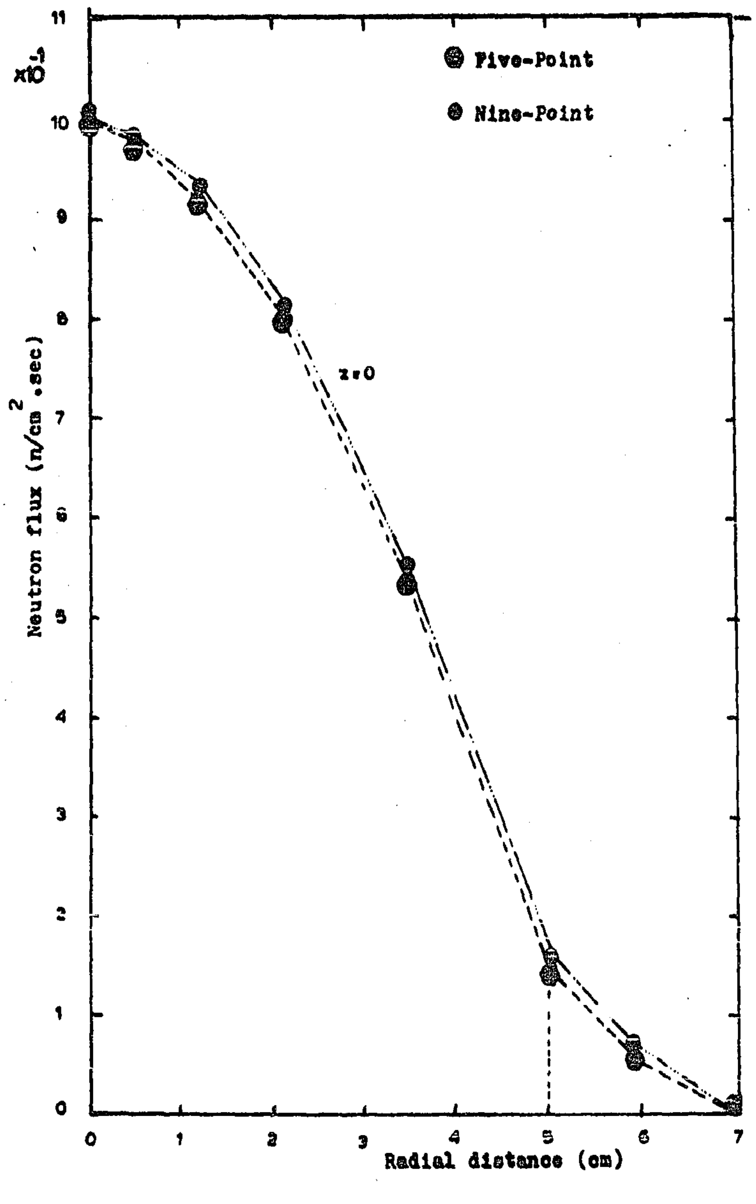


Fig. 68. Neutron flux as a function of radial distance for z = 0 and unequal spacing in a completely reflected cylindrical reactor core with more spacings along the radial axis

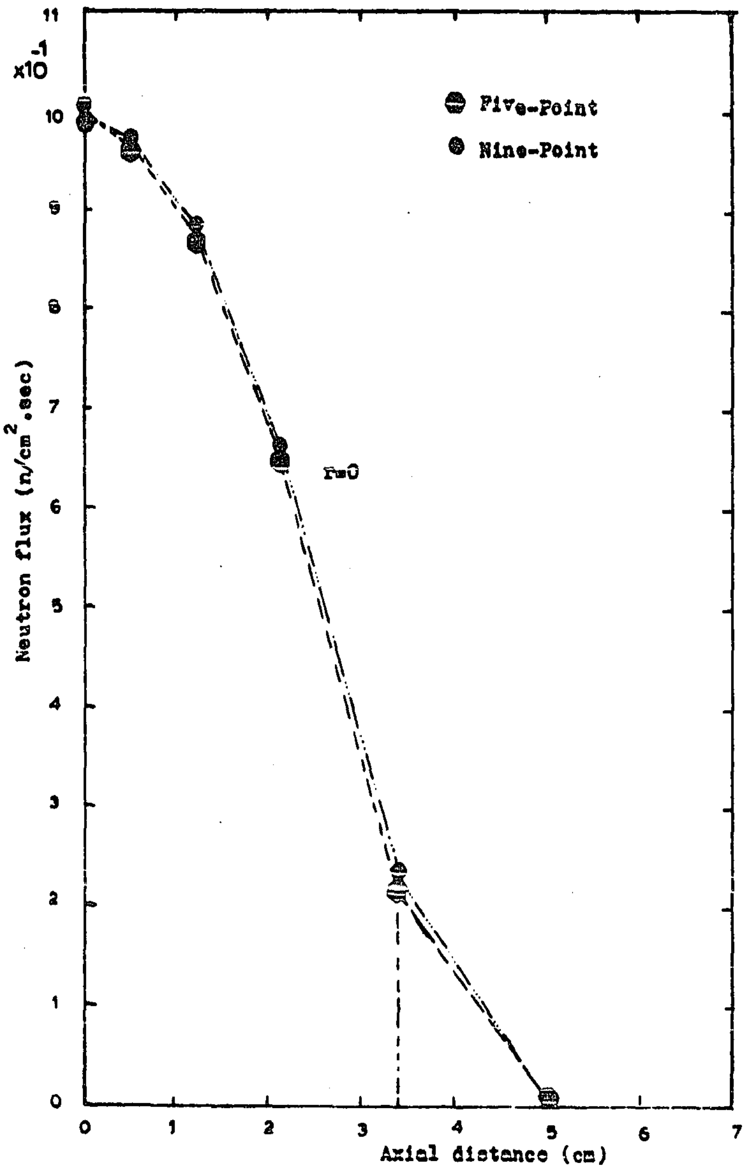


Fig. 69. Neutron flux as a function of axial distance for r = 0 and unequal spacing in a completely reflected cylindrical reactor core

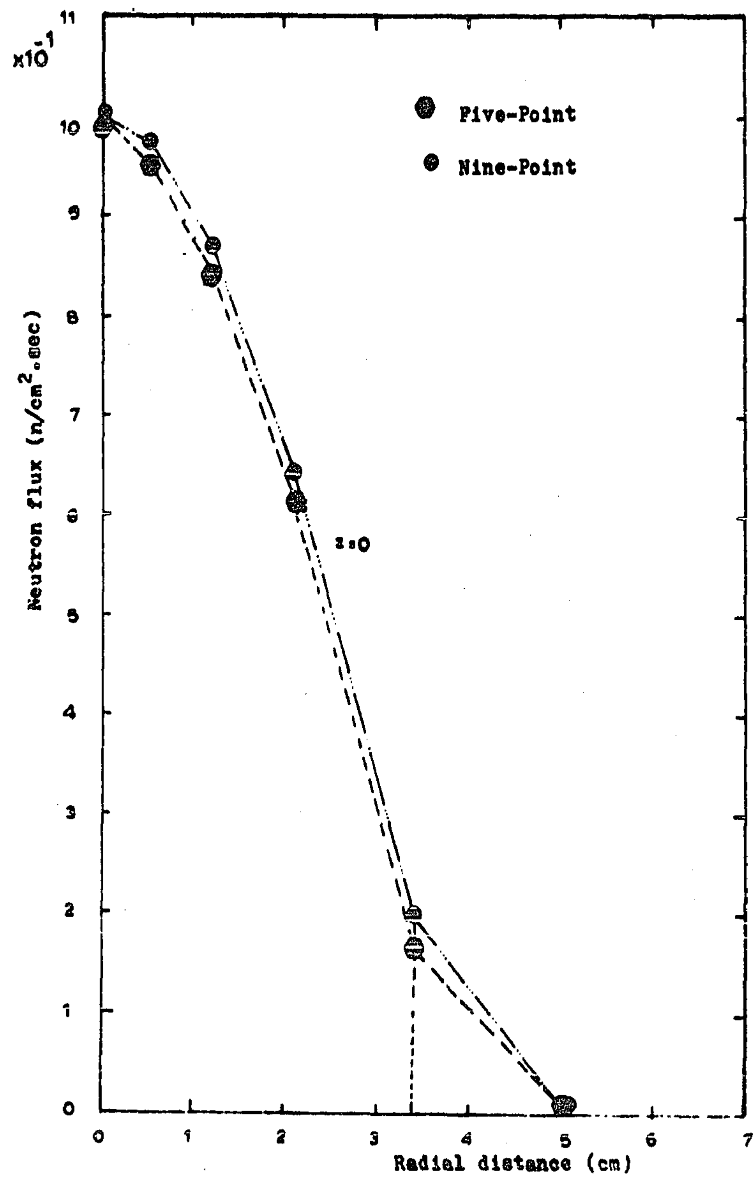


Fig. 70. Neutron flux as a function of radial distance for $z = 0$ and unequal spacing in a completely reflected cylindrical reactor core

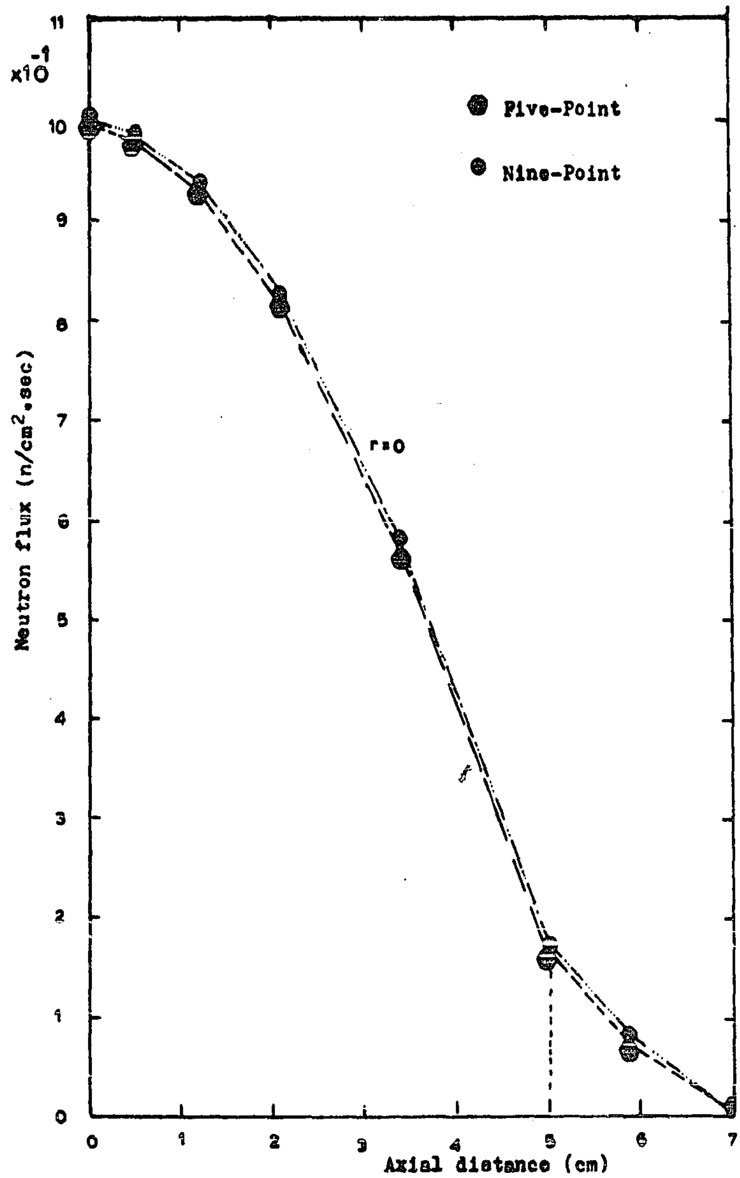


Fig. 71. Neutron flux as a function of axial distance for r = 0 and unequal spacing in a completely reflected cylindrical reactor core with more spacings along the axial axis

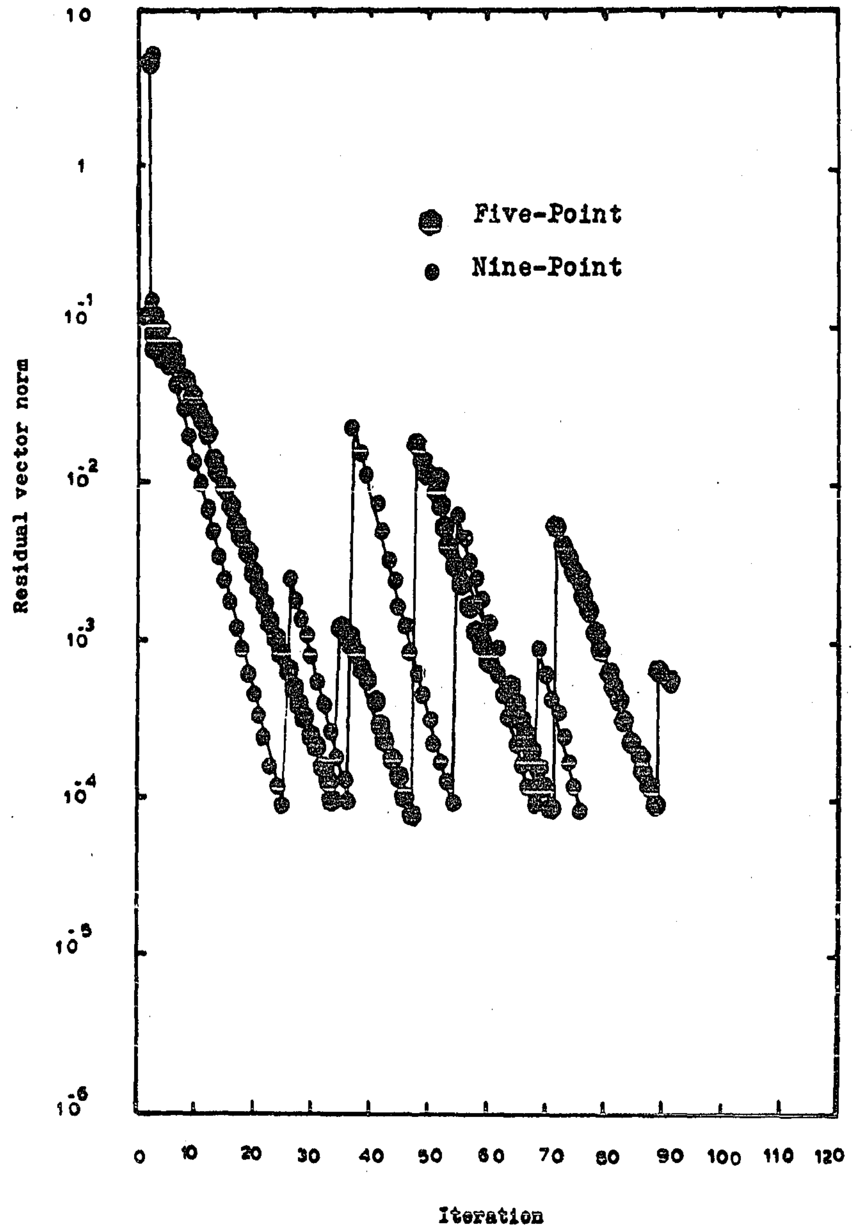


Fig. 72. Residual vector norm as a function of iteration and unequal spacing in a completely reflected cylindrical reactor core with more spacings along the radial axis

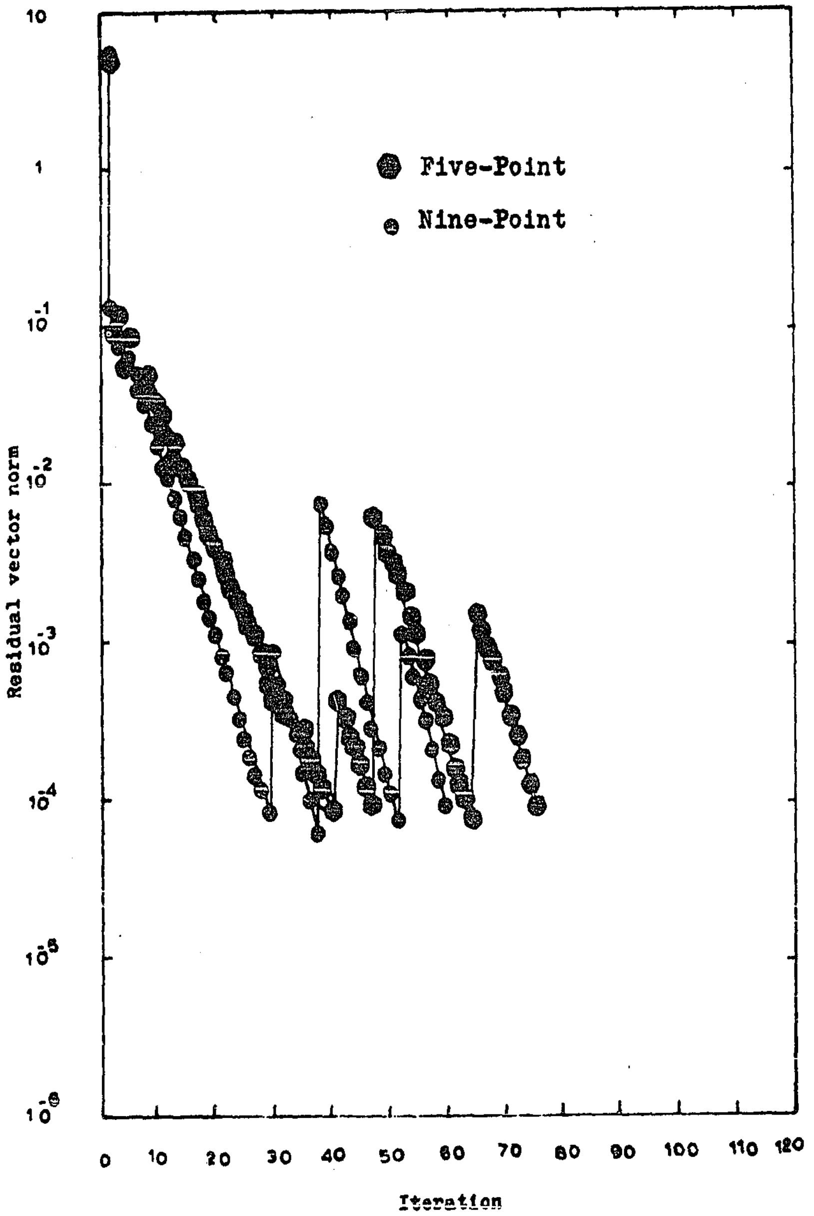


Fig. 73. Residual vector norm as a function of iteration and unequal spacing in a completely reflected cylindrical reactor core with more spacings along the axial axis

# IV. CONCLUSIONS

The technique of formulating the one-group time independent neutron diffusion equation in cylindrical geometry in terms of the nine-point finite difference equations resulted in more accuracy than the five-point formulation. The accuracy of the nine-point formulation over the five-point formulation was illustrated for a variety of configurations in reactor physics problems. The configurations include a bare reactor core, radially reflected reactor core, and axially reflected reactor core. The accuracy of the nine-point formula over the five-point formula is not affected by the spacing along the radial and axial axes. This accuracy was measured by the absolute error and Euclidean norm criteria.

The nine-point formula converges faster than the five-point formula for all calculations concerned with this reactor physics problem. This was illustrated by computing the slope of the curve from the plots of residual vector norm vs. iteration.

# V. SUGGESTION FOR FURTHER STUDY

The nine-point finite difference formulation technique for the one-group static neutron diffusion equation in two spatial dimensions (r-z) cylindrical geometry exhibited increased accuracy and convergence rate over the five-point finite difference formulation for regions with homogeneous Dirichlet boundary conditions for physical surface ($\phi_{surface}$ = 0) and homogeneous Neumann condition $\left(\frac{\partial\phi}{\partial r} = 0, \frac{\partial\phi}{\partial z} = 0\right)$ on the other boundaries. The nine-point finite difference formulation may be readily extended to the neutron diffusion equation in three dimensions (x, y, z) in rectangular geometry.

The present work may be extended to related fields dealing with the Helmholtz equation in r-z cylindrical geometry (for example the nonhomogeneous equation ($\nabla\cdot D(r,z)\nabla\phi(r,z)$ - $\Sigma(r,z)\phi(r,z) + S = 0$). Further work may be accomplished with the mixed boundary conditions which stated in physical terms means no inbound neutron current (or possibly an albedo boundary).

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## VII. ACKNOWLEDGMENTS

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# VIII. APPENDIX A: TAYLOR SERIES EXPANSIONS OF THE FLUX ABOUT A POINT $(r_i, z_j)$

1. The expansions of the flux at the points $(r_{i+1}, z_j)$, $(r_{i-1}, z_j)$, $(r_i, z_{j+1})$, and $(r_i, z_{j-1})$ in terms of $(r_i, z_j)$ are

$$\phi_{i+1} \equiv \phi_1$$

$$\phi_1 = \phi_o + RD_r\phi_o + \frac{R^2}{2} D_r^2\phi_o + \frac{R^3}{6} D_r^3\phi_o + O(h^4) \qquad \text{(A-1)}$$

$$\phi_{i,j-1} \equiv \phi_3$$

$$\phi_3 = \phi_o - L\, D_r\phi_o + \frac{L^2}{2} D_r^2\phi_o - \frac{L^3}{6} D_r^3\phi_o + O(h^4) \qquad \text{(A-2)}$$

$$\phi_{i,j+1} \equiv \phi_2$$

$$\phi_2 = \phi_o + T\, D_z\phi_o + \frac{T^2}{2} D_z^2\phi_o + \frac{T^3}{6} D_z^3\phi_o + O(h^4) \qquad \text{(A-3)}$$

$$\phi_{i,j-1} \equiv \phi_4$$

$$\phi_4 = \phi_o - B\, D_z\phi_o + \frac{B^2}{2} D_z^2\phi_o - \frac{B^3}{6} D_z^3\phi_o + O(h^4) \qquad \text{(A-4)}$$

2. The expansions of the flux at the corner point, (i+1,j+1), (i-1,j+1), (i-1,j-1), and (i+1,j-1) about the point (i,j) are

$$\phi_{i+1,j+1} \equiv \phi^1$$

$$\phi^1 = \sum_{m=0}^{3} \frac{1}{m!} (RD_r + TD_z)^m \phi_o$$

$$\begin{aligned}\phi^1 = \phi_o &+ RD_r\phi_o + TD_z\phi_o \\ &+ \frac{1}{2} R^2D_r^2\phi_o + \frac{1}{2} T^2D_z^2\phi_o + RTD_{rz}\phi_o \\ &+ \frac{1}{6} R^3D_r^3\phi_o + \frac{1}{6} T^3D_z^3\phi_o + \frac{1}{2} R^2TD_{rrz}\phi_o \\ &+ \frac{1}{2} RT^2D_{rzz}\phi_o \\ &+ O(h^4) \qquad \text{(A-5)}\end{aligned}$$

By adding and subtracting $\phi_o$ to the right hand side of Eq. 5 and by substituting for the resulted terms from Eq. A-1 and Eq. A-3, the result is

$$\begin{aligned}\phi_{i+1,j+1} = \phi_{i+1,j} + \phi_{i,j+1} - \phi_o &+ RTD_{rz}\phi_o \\ &+ \frac{1}{2} R^2TD_{rrz}\phi_o \\ &+ \frac{1}{2} RT^2D_{rzz}\phi_o \\ &+ O(h^4) \qquad \text{(A-6)}\end{aligned}$$

Eq. A-6 may be written as

$$\phi^1 - \phi_1 - \phi_2 + \phi_o = RTD_{rz}\phi_o + \frac{1}{2} R^2TD_{rrz}\phi_o + \frac{1}{2} RT^2D_{rzz}\phi_o + O(h^4) \qquad \text{(A-7)}$$

$$\phi_{i-1,j+1} \equiv \phi^2$$

$$\phi^2 = \sum_{m=0}^{3} \frac{1}{m!} (-LD_r + TD_z)^m \phi_o$$

$$\phi^2 = \phi_o - LD_r\phi_o + TD_z\phi_o$$

$$+ \frac{1}{2} L^2D_r^2\phi_o + \frac{1}{2} TD_z^2\phi_o - LTD_{rz}\phi_o$$

$$- \frac{1}{6} L^3D_r^3\phi_o + \frac{1}{6} T^3D_z^3\phi_o + \frac{1}{2} L^2TD_{rrz}\phi_o - \frac{1}{2} LT^2D_{rzz}\phi_o$$

$$+ O(h^4) \tag{A-8}$$

By adding and subtracting $\phi_o$ to the right hand side of Eq. A-8 and by substituting for the resultant terms from Eq. A-2 and Eq. A-3, the result is

$$\phi_{i-1,j+1} = \phi_{i-1,j} + \phi_{i,j+1} - \phi_o - LTD_{r_z}\phi_o$$

$$+ \frac{1}{2} L^2TD_{rrz}\phi_o$$

$$- \frac{1}{2} LT^2D_{rzz}\phi_o$$

$$+ O(h^4) \tag{A-9}$$

Equation A-9 may be expressed as

$$\phi^2 - \phi_3 - \phi_2 + \phi_o = - LTD_{rz}\phi_o + \frac{1}{2} L^2TD_{rrz}\phi_o - \frac{1}{2} LT^2D_{rzz}\phi_o$$

$$+ O(h^4) \tag{A-10}$$

$$\phi_{i-1,j-1} = \phi^3$$

$$\phi^3 = \sum_{m=0}^{3} \frac{1}{m!} (-LD_r - BD_z)^m \phi_o$$

$$\phi^3 = \phi_o - LD_{r\ o} - BD_{z\ o}$$

$$+ \frac{1}{2} L^2 D_r^2 \phi_o + \frac{1}{2} B^2 D_z^2 \phi_o + LBD_{rz} \phi_o$$

$$- \frac{1}{6} L^3 D_r^3 \phi_o - \frac{1}{6} D_z^3 \phi_o - \frac{1}{2} L^2 BD_{rz} \phi_o - \frac{1}{2} LB^2 D_{rzz} \phi_o$$

$$+ O(h^4) \qquad \text{(A-11)}$$

By adding and subtracting $\phi_o$ to the right hand side of Eq. A-11 and by substituting for the resultant terms from Eq. A-2 and Eq. A-4, the result is

$$\phi^3 = \phi_{i-1,j} + \phi_{i,j-1} - \phi_o + LBD_{r_z} \phi_o - \frac{1}{2} L^2 BD_{rrz} \phi_o$$

$$- \frac{1}{2} LB^2 D_{rzz} \phi_o + O(h^4) \qquad \text{(A-12)}$$

Equation A-12 may be written as

$$\phi^3 - \phi_3 - \phi_4 + \phi_o = LBD_{rz} \phi_o - \frac{1}{2} L^2 BD_{rrz} \phi_o - \frac{1}{2} LB^2 D_{rzz} \phi_o$$

$$+ O(h^4) \qquad \text{(A-13)}$$

$$\phi_{i+1,j+1} \equiv \phi^4$$

$$\phi^4 = \sum_{m=o}^{3} \frac{1}{m!} (RD_r - BD_z)^m \phi_o$$

$$\phi^4 = \phi_o + RD_r\phi_o - BD_z\phi_o$$

$$+ \frac{1}{2} R^2D_r^2\phi_o + \frac{1}{2} B^2D_z^2\phi_o - RBD_{rz}\phi_o$$

$$+ \frac{1}{6} R^3D_r^3\phi_o - \frac{1}{6} B^3D_z^3\phi_o - \frac{1}{2} R^2BD_{rrz}\phi_o + \frac{1}{2} RB^2D_{rzz}\phi_o$$

$$+ O(h^4) \qquad \text{(A-14)}$$

By adding and subtracting $\phi_o$ to Eq. A-14 and by substituting for the resultant terms from Eq. A-1 and Eq. A-3, the result is

$$\phi^4 - \phi_1 - \phi_4 + \phi_o = -RBD_{rz}\phi_o - \frac{1}{2} R^2BD_{rrz}\phi_o + \frac{1}{2} RB^2D_{rzz}\phi_o$$

$$+ O(h^4) \qquad \text{(A-15)}$$

# IX. APPENDIX B: SIMPLIFICATION OF DERIVATIVE TERMS

1. Simplification of the partial derivatives $\frac{\partial\phi}{\partial r}$ and $\frac{\partial\phi}{\partial z}$ at the points $(r_i + \frac{R}{2}, z-z_j)$, $(r_i - \frac{L}{2}, z-z_j)$, $(r-r_i, z_j + \frac{T}{2})$, and $(r-r_i, z_j - \frac{B}{2})$.

a. From Eq. 12

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i+\frac{R}{2}} = D_r\phi_o + \frac{R}{2} D_r^2\phi_o + (z-z_j)D_{rz}\phi_o + \frac{R^2}{8} D_r^2\phi_o$$

$$+ \frac{R}{2} (z-z_j)D_{rrz}\phi_o + \frac{1}{2} (z-z_j)^2 D_{rzz}\phi_o$$

Let $$A = D_r\phi_o + \frac{R}{2} D_r^2\phi_o + \frac{R^2}{8} D_r^3\phi_o$$

or $$A = \frac{1}{R} (RD_r\phi_o + \frac{R^2}{2} D_r^2\phi_o + \frac{R^3}{8} D_r^3\phi_o)$$

By adding and subtracting $\phi_o$ and $\frac{R^3}{6} D_r^3\phi_o$ to the right side, the result is

$$A = \frac{1}{R} (\phi_o + RD_r\phi_o + \frac{R^2}{2} D_r^2\phi_o + \frac{R^3}{6} D_r^3\phi_o$$

$$- \phi_o + \frac{R^3}{8} D_r^3\phi_o - \frac{R^3}{6} D_r^3\phi_o)$$

From Eq. A-1

$$\phi_1 = \phi_o + RD_r\phi_o + \frac{R^2}{2} D_r^2\phi_o + \frac{R^3}{6} D_r^3\phi_o + O(h^4)$$

thus,

$$A = \frac{\phi_1 - \phi_o}{R} - \frac{R^2}{24} D_r^2 \phi_o \qquad \text{(B-1)}$$

Let $\quad F = D_{r_z} + \frac{R}{2} D_{rrz}\phi_o$

From Eq. A-15

$$\phi^4 - \phi_1 - \phi_4 + \phi_o = -RBD_{rz}\phi_o - \frac{1}{2} R^2 BD_{rrz}\phi_o + \frac{1}{2} RB^2 D_{rzz}\phi_o + O(h^4)$$

Thus,

$$F = - \frac{\phi^4 - \phi_1 - \phi_4 + \phi_o}{RB} + \frac{B}{2} D_{rzz}\phi_o \qquad \text{(B-2)}$$

From Eq. A-7

$$\phi^1 - \phi_1 - \phi_2 + \phi_o = RTD_{rz}\phi_o + \frac{1}{2} R^2 TD_{rrz}\phi_o + \frac{1}{2} RT^2 D_{r_{zz}}\phi_o + O(h^4)$$

Thus,

$$F = \frac{\phi^1 - \phi_1 - \phi_2 + \phi_o}{RT} - \frac{T}{2} D_{rzz}\phi_o \qquad \text{(B-2')}$$

Let

$$C = \frac{1}{2} D_{rzz}\phi_o \qquad \text{(B-3)}$$

Therefore, Eq. 12 may be written as

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i+\frac{R}{2}} = A + (z-z_j)F + (z-z_j)^2C \qquad \text{(B-4)}$$

b. From Eq. 13

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=r_i-\frac{L}{2}} = D_r\phi_o - \frac{L}{2} D_r^2\phi_o + (z-z_j)D_{rz}\phi_o + \frac{L^2}{8} D_r^3\phi_o$$

$$- \frac{L}{2} (z-z_j)D_{r_{rz}}\phi_o + \frac{1}{2} (z-z_j)^2 D_{rzz}\phi_o$$

Let $$A' = D_r\phi_o - \frac{L}{2} D_r^2\phi_o + \frac{L^2}{8} D_r^3\phi_o$$

or $$A' = -\frac{1}{L} (-LD_r\phi_o + \frac{L^2}{2} D_r^2\phi_o - \frac{L^3}{8} D_r^3\phi_o)$$

By adding and subtracting $\phi_o$ and $\frac{L^3}{6} D_r^3\phi_o$ to the right side, the result is

$$A' = -\frac{1}{L} (\phi_o - LD_r\phi_o + \frac{L^2}{2} D_r^2\phi_o - \frac{L^3}{6} D_r^3\phi_o + \frac{L^3}{6} D_r^3\phi_o$$

$$- \frac{L^3}{8} D_r^3\phi_o - \phi_o)$$

From Eq. A-2

$$\phi_3 = \phi_o - LD_r\phi_o + \frac{L^2}{2} D_r^2\phi_o - \frac{L^3}{6} D_r^3\phi_o + O(h^4)$$

Thus,

$$A' = -\frac{\phi_3 - \phi_o}{L} - \frac{L^2}{24} D_r^3 \phi_o \qquad \text{(B-5)}$$

Let

$$F' = D_{rz}\phi_o - \frac{L}{2} D_{rrz}\phi_o$$

From Eq. A-10

$$\phi^2 - \phi_3 - \phi_2 + \phi_o = - LTD_{rz}\phi_o + \frac{1}{2} L^2 TD_{rrz}\phi_o - \frac{1}{2} LT^2 D_{rzz}\phi_o$$

$$+ O(h^4)$$

Thus,

$$F' = -\frac{\phi^2 - \phi_3 - \phi_2 + \phi_o}{LT} - \frac{T}{2} D_{rzz}\phi_o \qquad \text{(B-6)}$$

From Eq. A-13

$$\phi^3 - \phi_3 - \phi_4 + \phi_o = LBD_{rz}\phi_o - \frac{1}{2} L^2 BD_{rrz}\phi_o - \frac{1}{2} LB^2 D_{rzz}\phi_o$$

$$+ O(h^4)$$

Thus,

$$F' = \frac{\phi^3 - \phi_3 - \phi_4 + \phi_o}{LB} + \frac{1}{2} BD_{rzz}\phi_o \qquad \text{(B-6')}$$

Therefore, Eq. 13 may be written as

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=r_i-\frac{L}{2}} = A' + (z-z_j)F' + (z-z_j)^2 C \qquad \text{(B-7)}$$

c. From Eq. 14

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=z_j+\frac{T}{2}} = D_z\phi_o + (r-r_i)D_{rz}\phi_o + \frac{T}{2} D_z^2\phi_o$$

$$+ \frac{1}{2} (r-r_i)^2 D_{rrz}\phi_o$$

$$+ \frac{T}{2} (r-r_i)D_{rzz}\phi_o + \frac{T^2}{8} D_z^3\phi_o$$

Let $\quad G = D_z\phi_o + \frac{T}{2} D_z^2\phi_o + \frac{T^2}{8} D_z^3\phi_o$

or $\quad G = \frac{1}{T} (TD_z\phi_o + \frac{T^2}{2} D_z^2\phi_o + \frac{T^3}{8} D_z^3\phi_o)$

By adding and subtracting $\phi_o$ and $\frac{T^3}{6} D_z^3\phi_o$ to the right side, the result is

$$G = \frac{1}{R} (\phi_o + TD_z\phi_o + \frac{T^2}{2} D_z^2\phi_o + \frac{T^3}{6} D_z^3\phi_o + \frac{T^3}{8} D_z^3\phi_o - \frac{T^3}{8} D_z^3\phi_o - \phi_o)$$

From Eq. A-3

$$\phi_2 = \phi_o + TD_z\phi_o + \frac{T^2}{2} D_z^2\phi_o + \frac{T^3}{6} D_z^3\phi_o + O(h^4)$$

Thus,

$$G = \frac{\phi_2 - \phi_o}{T} - \frac{T^2}{24} D_z^3\phi_o \qquad \text{(B-8)}$$

Let $$H = D_{rz}\phi_o + \frac{T}{2} D_{rzz}\phi_o$$

From Eq. A-7

$$\phi^1 - \phi_1 - \phi_2 + \phi_o = RTD_{rz}\phi_o + \frac{1}{2} R^2TD_{rrz}\phi_o + \frac{1}{2} RT^2D_{rzz}\phi_o$$

$$+ O(h^4)$$

Thus,

$$H = \frac{\phi^1 - \phi_1 - \phi_2 + \phi_o}{RT} - \frac{1}{2} R\, D_{rrz}\phi_o \qquad \text{(B-9)}$$

From Eq. A-10

$$\phi^2 - \phi_3 - \phi_2 + \phi_o = - LTD_{rz}\phi_o + \frac{1}{2} L^2TD_{rrz}\phi_o - \frac{1}{2} LT^2D_{rzz}\phi_o$$

$$+ O(h^4)$$

Thus,

$$H = - \frac{\phi^2 - \phi_3 - \phi_2 + \phi_o}{LT} + \frac{1}{2} L\, D_{rrz}\phi_o \qquad \text{(B-9')}$$

Let $$K = \frac{1}{2} D_{rrz}\phi_o \qquad \text{(B-10)}$$

Therefore, Eq. 14 may be written as

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=z_j+\frac{T}{2}} = G + (r-r_i)\, H + (r-r_i)^2\, K \qquad \text{(B-11)}$$

d. From Eq. 15

$$\left.\frac{\partial\phi}{\partial z}\right|_{z-z_j-\frac{B}{2}} = D_z\phi_o + (r-r_i)D_{rz}\phi_o - \frac{B}{2} D_z^2\phi_o + \frac{1}{2} (r-r_i)^2 D_{rrz}\phi_o$$

$$- \frac{B}{2} (r-r_i)D_{rzz}\phi_o + \frac{B^2}{8} D_z^3\phi_o$$

Let

$$G' = D_z\phi_o - \frac{B}{2} D_z^2\phi_o + \frac{B^2}{8} D_z^3\phi_o$$

or

$$G' = - \frac{1}{B} (-B\ D_z\phi_o + \frac{B^2}{2} D_z^2\phi_o - \frac{B^3}{8} D_z^3\phi_o)$$

By adding and subtracting $\phi_o$ and $\frac{B^3}{6} D_z^3\phi_o$ to the right side, the result is

$$G' = - \frac{1}{B} (\phi_o - BD_z\phi_o + \frac{B^2}{2} D_z^2\phi_o - \frac{B^3}{6} D_z^3\phi_o - \phi_o$$

$$+ \frac{B^3}{6} D_z^3\phi_o - \frac{B^3}{8} D_z^3\phi_o)$$

From Eq. A-4

$$\phi_4 = \phi_o - BD_z\phi_o + \frac{B^2}{2} D_z^2\phi_o - \frac{B^3}{6} D_z^3\phi_o + O(h^4)$$

Thus,

$$G' = - \frac{\phi_4 - \phi_o}{B} - \frac{1}{24} B^2 D_z^3\phi_o \qquad \text{(B-12)}$$

Let

$$H' = D_{r_z}\phi_o - \frac{B}{2} D_{rzz}\phi_o$$

$$H' = \frac{1}{LB} (LBD_r \ \phi_o - \frac{1}{2} LB^2 D_{rzz}\phi_o)$$

From Eq. A-13

$$\phi^3 - \phi_3 - \phi_4 + \phi_o = LBD_{rz}\phi_o - \frac{1}{2} L^2 BD_{rrz}\phi_o - \frac{1}{2} LB^2 D_{rzz}\phi_o$$

$$+ O(h^4)$$

Thus,

$$H' = \frac{\phi^3 - \phi_3 - \phi_4 + \phi_o}{LB} + \frac{1}{2} L \ D_{rrz}\phi_o \qquad \text{(B-13)}$$

From Eq. A-15

$$\phi^4 - \phi_1 - \phi_4 + \phi_o = - RBD_{rz}\phi_o - \frac{1}{2} R^2 BD_{rrz}\phi_o$$

$$+ \frac{1}{2} RB^2 D_r \ \ \phi_o + O(h^4)$$

Thus,

$$H' = - \frac{\phi^4 - \phi_1 - \phi_4 + \phi_o}{RB} - \frac{1}{2} R \ D_{rrz}\phi_o \qquad \text{(B-13')}$$

Therefore, Eq. 15 may be written as

$$\left.\frac{\partial\phi}{\partial r}\right|_{z=z_j-\frac{B}{2}} = G' + (r-r_i)H' + (r-r_i)^2 \ K \qquad \text{(B-14)}$$

2. Simplification of P, Q, and M.

a:

$$P = r_i^2 - (r_i + \frac{R}{2})^2$$

$$P = r_i^2 - r_i^2 - r_i R - \frac{R^2}{4}$$

$$P = - r_i R - \frac{R^2}{4} \qquad \text{(B-15)}$$

b:

$$Q = r_i^3 - (r_i + \frac{R}{2})^3$$

$$Q = r_i^3 - r_i^3 - 3r_i^3 \frac{R}{2} - 3r_i \frac{R^2}{4} - \frac{R^3}{4}$$

$$Q = - \frac{3}{2} r_i^2 R - \frac{3}{4} r_i R^2 - \frac{R^3}{8} \qquad \text{(B-16)}$$

c:

$$M = r_i^4 - (r_i + \frac{R}{2})^4$$

$$M = r_i^4 - r_i^4 - 4r_i^3 \frac{R}{2} - 6r_i^2 \frac{R}{24} - 4r_i \frac{R^3}{8} - \frac{R^4}{16}$$

$$M = -2r_i^3 R - \frac{3}{2} r_i^2 R^2 - \frac{1}{2} r_i R^3 - \frac{R^4}{16} \qquad \text{(B-17)}$$

c: Evaluation of $\frac{1}{3} Q - \frac{1}{2} r_i P$

By substituting for P and Q from Eq. B-15 and Eq. B-16, respectively, the result is

$$\frac{1}{3} Q - \frac{1}{2} r_i P = - \frac{1}{8} r_i R^2 - \frac{R^3}{24} \qquad \text{(B-18)}$$

d: Evaluation of $\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P$

By substituting for P, Q, and M from the Equations (B-15, B-16, and B-17), respectively, the result is

$$\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P = - \frac{1}{24} r_i R^3 - \frac{1}{64} R^4 \qquad \text{(B-19)}$$

e: Evaluation of $[R(\frac{1}{3} Q - \frac{1}{2} r_i P) - (\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P)]$

By substituting for $(\frac{1}{3} Q - \frac{1}{2} r_i P)$ from Eq. B-18 and for $(\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P)$ from Eq. B-19, the result is

$$R(\frac{1}{3} Q - \frac{1}{2} r_i P) = \frac{1}{8} r_i R^3 - \frac{1}{24} R^4 \qquad \text{(B-20)}$$

Thus,

$$R(\frac{1}{3} Q - \frac{1}{2} r_i P) - (\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P) = - \frac{1}{12} r_i R^3 - \frac{1}{96} R^4 \qquad \text{(B-21)}$$

3. Evaluation

a:

$$\frac{1}{2} D_1 [R(\frac{1}{3} Q - \frac{1}{2} r_i P) - (\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P)] D_{rrz} \phi_o$$

By substituting for $[R(\frac{1}{3} Q - \frac{1}{2} r_i P) - (\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P)]$ from Eq. B-21 and lumping the term $- \frac{1}{92} D_1 R^4 D_{rrz} \phi_o$ with the truncation error $O(h^4)$, one has

$$\frac{1}{2} D_1 [R(\frac{1}{3} Q - \frac{1}{2} r_i P) - (\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P)] = - \frac{1}{24} D_1 r_i R^3 D_{rrz} \phi_o \qquad \text{(B-22)}$$

b:

$$\frac{1}{48} D_1 P T^2 \, D_z^3 \phi_o$$

By substituting for P from Eq. B-15 and then lumping the term $- \frac{1}{192} D_1 T^2 R^2 D_z^3 \phi_o$ with the truncation error $O(h^4)$ one has

$$\frac{1}{48} D_1 P T^2 D_z^3 \phi_o = - \frac{1}{48} D_1 r_i R T^2 D_z^3 \phi_o \qquad \text{(B-23)}$$

Thus, the terms that include the derivatives in Eq. 17 may be written as

$$\begin{aligned} ID = &+ \frac{1}{2} D_1 [R(\frac{1}{3} Q - \frac{1}{2} r_i P) - (\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P)] D_{rrz} \phi_o \\ &+ \frac{1}{48} D_1 P T^2 D_z^3 \phi_o \\ &- \frac{1}{48} D_1 r_i R^3 T D_r^3 \phi_o \\ &- \frac{1}{24} D_1 r_i T^3 D_{rzz} \phi_o \end{aligned} \qquad \text{(B-24)}$$

By substituting for the first and second term in B-24 from Eq. B-22 and Eq. B-23, one has

$$\begin{aligned} ID = &- \frac{1}{24} D_1 r_i R^3 D_{rrz} \phi_o - \frac{1}{48} D_1 r_i R T^2 D_z^3 \phi_o \\ &- \frac{1}{48} D_1 r_i R^2 T D_r^3 \phi_o - \frac{1}{24} D_1 r_i T^3 D_{rzz} \phi_o \end{aligned}$$

or

$$\begin{aligned} ID = &- \frac{1}{48} D_1 r_i R T (T D_z^3 \phi_o + R D_r^3 \phi_o) \\ &- \frac{1}{24} (R^3 D_{rrz} \phi_o + T^3 D_{rzz} \phi_o) \end{aligned} \qquad \text{(B-25)}$$

4. Simplification of P', Q', and M'

a:

$$P' = (r_i - \frac{L}{2})^2 - r_i^2$$

$$P' = r_i^2 - r_i L + \frac{L^2}{4} - r_i^2$$

$$P' = -r_i L + \frac{L^2}{4} \qquad \text{(B-26)}$$

b:

$$Q' = (r_i - \frac{L}{2})^3 - r_i^3$$

$$Q' = r_i^3 - 3r_i^2 \frac{L}{2} + 3r_i \frac{L^2}{4} - \frac{L^3}{8} - r_i^3$$

$$Q' = -\frac{3}{2} r_i^2 L + \frac{3}{4} r_i L^2 - \frac{L^3}{8} \qquad \text{(B-27)}$$

c:

$$M' = (r_i - \frac{L}{2})^4 - r_i^4$$

$$M' = r_i^4 - 4r_i^3 \frac{L}{2} + 6r_i^2 \frac{L^2}{4} - 4r_i \frac{L^3}{8} + \frac{L^4}{16} - r_i^4$$

$$M' = -2r_i^3 L + \frac{3}{2} r_i^2 L^2 - \frac{1}{2} r_i L^3 + \frac{L^4}{16} \qquad \text{(B-28)}$$

d: Evaluation of $[(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i^2 P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')]$

By substituting for P', Q', and M' from the equations (B-26, B-27, and B-28) one has

$$\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i P' = -\frac{1}{24} r_i L^3 + \frac{L^4}{64}$$

(B-29)

$$L(\frac{1}{3} Q' - \frac{1}{2} r_i P') = \frac{1}{8} r_i L^3 - \frac{1}{24} L^4$$

or

$$[(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')]$$

$$= \frac{1}{12} r_i L^3 - \frac{1}{96} L^4 \tag{B-30}$$

5. Evaluation

a:

$$- \frac{1}{2} D_2[(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')] D_{rrz}\phi_o$$

By substituting for $[(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')]$ from Eq. B-30 and lumping the resulted term $\frac{1}{192} D_2 L^4 D_{rrz}\phi_o$ with the truncation error $O(h^4)$ one has

$$- \frac{1}{2} D_2[(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')] D_{rrz}\phi_o$$

$$= - \frac{1}{24} D_2 r_i L^3 D_{rrz}\phi_o \tag{B-31}$$

b:

$$\frac{1}{48} D_2 T^2 P' D_z^3 \phi_o$$

By substituting for P' from Eq. B-26 and lumping the resulted term $\frac{1}{192} D_2 L^2 T^2 D_z^3 \phi_o$ with the truncation error $O(h^4)$,

$$\frac{1}{48} D_2 T^2 P' D_z^3 \phi_o = - \frac{1}{48} D_2 r_i L T^2 D_z^3 \phi_o \tag{B-32}$$

Thus, the terms that include the derivatives in Eq. 19 may be written as

$$IE = -\frac{1}{24} D_2 r_i L^3 D_{rrz}\phi_o$$

$$-\frac{1}{48} D_2 r_i L T^2 D_z^3\phi_o$$

$$+\frac{1}{48} D_2 r_i L^2 T D_r^3\phi_o$$

$$+\frac{1}{24} D_2 r_i T^3 D_{rzz}\phi_o$$

or

$$IE = -\frac{1}{48} D_2 r_i LT(TD_z^3\phi_o - L D_r^3\phi_o)$$

$$-\frac{1}{24} D_2 r_i (L^3 D_{rrz}\phi_o - T^3 D_{rzz}\phi_o) \qquad \text{(B-33)}$$

c:

$$-\frac{1}{48} D_3 B^2 P' D_z^3\phi_o$$

By substituting for P' from Eq. B-26 and lumping the resulted term $-\frac{1}{192} D_3 B^2 L^2 D_z^3\phi_o$ with the truncation error $O(h^4)$, one has

$$-\frac{1}{48} D_3 B^2 P' D_z^3\phi_o = \frac{1}{48} D_3 r_i B^2 L D_z^3\phi_o \qquad \text{(B-34)}$$

d:

$$\frac{1}{2} D_3 [(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i^2 P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')] D_{rrz}\phi_o$$

By substituting for $[(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i^2 P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')]$ from Eq. B-31, one has

$$\frac{1}{2} D_3 [(\frac{1}{4} M' - \frac{2}{3} r_i Q' + \frac{1}{2} r_i^2 P') + L(\frac{1}{3} Q' - \frac{1}{2} r_i P')] D_{rrz} \phi_o$$

$$= \frac{1}{24} D_3 r_i L^3 D_{rrz} \phi_o \qquad \text{(B-35)}$$

Thus, the terms that include the derivatives in Eq. 21 may be written as

$$IG = \frac{1}{48} D_3 r_i B^2 L D_z^3 \phi_o + \frac{1}{48} D_3 r_i L^2 B D_r^3 \phi_o + \frac{1}{24} D_3 r_i L^3 D_{rrz} \phi_o + \frac{1}{24} D_3 r_i B^3 D_{rzz} \phi_o$$

or

$$IG = \frac{1}{48} D_3 r_i LB(BD_z^3 \phi_o + LD_r^3 \phi_o) + \frac{1}{24} D_3 r_i (L^3 D_{rrz} \phi_o + B^3 D_{rzz} \phi_o) \qquad \text{(B-36)}$$

e:

$$- \frac{1}{48} D_4 B^2 P D_z^3 \phi_o$$

By substituting for P from Eq. B-15 and lumping the resulted term $- \frac{1}{192} D_4 B^2 R^2 D_z^3 \phi_o$ with the truncation error $O(h^4)$, one has

$$- \frac{1}{48} D_4 B^2 P D_z^3 \phi_o = \frac{1}{48} D_3 r_i B^2 R D_z^3 \phi_o \qquad \text{(B-37)}$$

f:

$$\frac{1}{2} D_4 [(\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P) - R(\frac{1}{3} Q - \frac{1}{2} r_i P)] D_{rrz} \phi_o$$

By substituting for $[(\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P) - R(\frac{1}{3} Q - \frac{1}{2} r_i P)]$ from Eq. B-22, one has

$$\frac{1}{2} D_4 [(\frac{1}{4} M - \frac{2}{3} r_i Q + \frac{1}{2} r_i^2 P) - R(\frac{1}{3} Q - \frac{1}{2} r_i P)] D_{rrz} \phi_o$$

$$= \frac{1}{24} D_4 r_i R^3 D_{rrz} \phi_o \qquad \text{(B-38)}$$

Thus, the terms those include the derivatives in Eq. 23 may be written as

$$IH = \frac{1}{48} D_4 r_i B^2 R D_z^3 \phi_o$$

$$- \frac{1}{48} D_4 r_i B R^2 D_r^3 \phi_o$$

$$+ \frac{1}{24} D_4 r_i R^3 D_{r_{rz}} \phi_o$$

$$- \frac{1}{24} D_4 r_i B^3 D_{rzz} \phi_o$$

$$IH = \frac{1}{48} D_4 r_i R B (B D_z^3 \phi_o - R D_r^3 \phi_o)$$

$$+ \frac{1}{24} D_4 r_i (R^3 D_{rrz} \phi_o - B^3 D_{rzz} \phi_o) \qquad \text{(B-39)}$$

# X. APPENDIX C: APPLICATION OF THE IDENTITY $\nabla^2\phi = \frac{\Sigma_n}{D_n}\phi$

1. Simplification of the following terms:

a: $-\frac{1}{48} D_1 \, r_i \; RT(T \, D_z^3\phi_o + R \, D_r^3\phi_o)$

b: $-\frac{1}{48} D_2 \, r_i \; LT(T \, D_z^3\phi_o - L \, D_r^3\phi_o)$

c: $+\frac{1}{48} D_3 \, r_i \; LB(B \, D_z^3\phi_o + L \, D_r^3\phi_o)$

d: $+\frac{1}{48} D_4 \, r_i \; RB(B \, D_z^3\phi_o - R \, D_r^3\phi_o)$

The above terms have appeared in relations 25, 33, 36, and 39 of Appendix B, respectively. These terms may be further simplified by using the following techniques:

$$\nabla^2\phi_o - \sigma_n\phi = 0 \tag{C-1}$$

where $\sigma_n = \frac{\Sigma_n}{D_n}$, $n = 1,2,3,4$

Thus,

$$\nabla^2\phi_o = \sigma_n\phi_o \tag{C-2}$$

The Laplacian operator $\nabla^2$ may be expressed in cylindrical geometry (r-z coordinates) as

$$\nabla^2 = D_r^2 + D_z^2 + \frac{1}{r} D_r$$

or

$$\nabla^2\phi_o = (D_r^2 + D_z^2 + \frac{1}{r_i} D_r)\phi_o$$

a: $$-\frac{1}{48} D_1 r_i RT(TD_z^3\phi_o + RD_r^3\phi_o)$$

This could be simplified by using the following identity:

$$(RD_r + TD_z)\nabla^2\phi_o = (RD_r + TD_z)(D_r^2 + D_z^2 + \frac{1}{r_i} D_r)\phi_o$$

By substituting for $\nabla^2\phi_o$ from Eq. C-2, one has

$$(RD_r + TD_z)\sigma_1\phi_o = RD_r^3\phi_o + RD_{rzz} + \frac{R}{r_i} D_r^2\phi_o$$

$$+ TD_{rrz}\phi_o + TD_z^3\phi_o + \frac{T}{r_i} D_{rz}\phi_o$$

or

$$T D_z^3\phi_o + R D_r^3\phi_o = (R D_r + T D_z)\sigma_1\phi_o$$

$$- \frac{1}{r_i} (RD_r^2\phi_o + TD_{r_z}\phi_o)$$

$$- (RD_{rzz}\phi_o + TD_{rrz}\phi_o) \qquad \text{(C-3)}$$

By multiplying both sides of Eq. C-3 by $-\frac{1}{48} D_1 r_i RT$, one has

$$-\frac{1}{48} D_1 r_i RT(TD_z^3\phi_o + RD_r^3\phi_o) = -\frac{1}{48} D_1 r_i T(RD_r + TD_z)\sigma_1\phi_o$$

$$+ \frac{1}{48} D_1RT(RD_r^2\phi_o + TD_{rz}\phi_o)$$

$$+ \frac{1}{48} D_1r_iRT(RD_{rzz}\phi_o + TD_{rrz}\phi_o)$$

(C-4)

The term $-\frac{1}{48} D_1 r_i \sigma_1 RT(RD_r + TD_z)\phi_o$ that appeared in Eq. C-4 may be simplified using the following technique:

From Eq. A-1 and Eq. A-3

$$\phi_1 = \phi_o + RD_r\phi_o + \frac{R^2}{2} D_r^2\phi_o + \frac{R^3}{6} D_r^3\phi_o + O(h^4)$$

$$\phi_2 = \phi_o + TD_z\phi_o + \frac{T^2}{2} D_z^2\phi_o + \frac{T^3}{6} D_z^3\phi_o + O(h^4)$$

By adding the equations then multiplying both sides of resulted equation by $-\frac{1}{48} D_1 r_i RT \sigma_1$ and then lumping the terms of orders $O(h^4)$ and higher with the truncation error $O(h^4)$, one has

$$-\frac{1}{48} D_1 r_i \sigma_1 RT(RD_r\phi_o + TD_z\phi_o) = -\frac{1}{48} D_1 r_i \sigma_1 RT(\phi_1+\phi_2-2\phi_o) + O(h^4) \quad \text{(C-5)}$$

By substituting for $-\frac{1}{48} D_1 r_i \sigma_1 RT (RD_r\phi_o + TD_z\phi_o)$ in Eq. C-4 from Eq. C-5, one has

$$-\frac{1}{48} D_1 r_i RT(TD_z^3 + RD_r^3\phi_o) = -\frac{1}{48} D_1 r_i \sigma_1 RT(\phi_1 + \phi_2 - 2\phi_o) + \frac{1}{48} D_1 RT(RD_r^2\phi_o + TD_{rz}\phi_o) + \frac{1}{48} D_1 r_i RT(RD_{rzz}\phi_o + TD_{rrz}\phi_o) + O(h^4) \quad \text{(C-6)}$$

Using the same technique as in (a) the terms in (b), (c), and (d) may be simplified. Thus,

b:

$$- \frac{1}{48} D_2 r_i LT(TD_z^3\phi_o - LD_r^3\phi_o) = - \frac{1}{48} D_2 r_i \sigma_2 LT(\phi_2 + \phi_3 - 2\phi_o)$$

$$+ \frac{1}{48} D_2 LT(Td_{r_z}\phi_o - LD_r^2\phi_o)$$

$$+ \frac{1}{48} D_2 r_i LT(TD_{rrz}\phi_o - LD_{rzz}\phi_o)$$

$$+ O(h^4) \qquad \text{(C-7)}$$

c:

$$+ \frac{1}{48} D_3 r_i LB(BD_z^3\phi_o + LD_r^3\phi_o) = - \frac{1}{48} D_3 r_i \sigma_3 LB(\phi_4 + \phi_3 - 2\phi_o)$$

$$- \frac{1}{48} D_3 LB(BD_{rz}\phi_o + LD_r^2\phi_o)$$

$$- \frac{1}{48} D_3 r_i LB(BD_{rrz}\phi_o + LD_{rzz}\phi_o)$$

$$+ O(h^4) \qquad \text{(C-8)}$$

d:

$$\frac{1}{48} D_4 r_i RB(BD_z^3\phi_o - RD_r^3\phi_o) = - \frac{1}{48} D_4 r_i \sigma_4 RB(\phi_4 + \phi_1 - 2\phi_o)$$

$$- \frac{1}{48} D_4 RB(BD_{r_z}\phi_o - RD_r^2\phi_o)$$

$$- \frac{1}{48} D_4 r_i RB(BD_{rrz}\phi_o - RD_{rzz}\phi_o)$$

$$+ O(h^4) \qquad \text{(C-9)}$$

By adding Eq. C-6, C-7, C-8, and C-9, one has

$$- \frac{1}{48} D_1 r_i RT(TD_z^3\phi_o + RD_r^3\phi_o)\Big|_{R_1} - \frac{1}{48} D_2 r_i LT(TD_z^3\phi_o - LD_r^3\phi_o)\Big|_{R_2}$$

$$+ \frac{1}{48} D_3 r_i LB(BD_z^3\phi_o + LD_r^3\phi_o)\Big|_{R_3} + \frac{1}{48} D_4 r_i RB(BD_z^3\phi_o - RD_r^3\phi_o)\Big|_{R_4}$$

$$= - \frac{1}{48} D_1\sigma_1\ r_i\ RT(\phi_1 + \phi_2 - 2\phi_o)$$

$$- \frac{1}{48} D_2\sigma_2\ r_i\ LT(\phi_2 + \phi_3 - 2\phi_o)$$

$$- \frac{1}{48} D_3\sigma_3\ r_i\ LB(\phi_4 + \phi_3 - 2\phi_o)$$

$$- \frac{1}{48} D_4\sigma_4\ r_i\ RB(\phi_4 + \phi_1 - 2\phi_o)$$

$$+ \frac{1}{48} D_1\ RT(RD_r^2\phi_o + TD_{rz}\phi_o)\Big|_{R_1}$$

$$+ \frac{1}{48} D_2\ LT(TD_{r_z}\phi_o - LD_r^2\phi_o)\Big|_{R_2}$$

$$- \frac{1}{48} D_3\ LB(BD_{r_z}\phi_o + LD_r^2\phi_o)\Big|_{R_3}$$

$$- \frac{1}{48} D_4\ RB(BD_{r_z}\phi_o - RD_r^2\phi_o)\Big|_{R_4}$$

$$+ \frac{1}{48} D_1 r_i RT(RD_{rzz}\phi_o + TD_{rrz}\phi_o)\Big|_{R_1}$$

$$+ \frac{1}{48} D_2 r_i LT(TD_{rrz}\phi_o - LD_{rzz}\phi_o)\Big|_{R_2}$$

$$- \frac{1}{48} D_3 r_i LB(BD_{rrz}\phi_o + LD_{rzz}\phi_o) - \frac{1}{48} D_4 r_i RB(BD_{rrz}\phi_o - RD_{rrz}\phi_o)\Big|_{R_4}$$

(C-10)

The terms including the derivatives in Eq. C-10 may be rearranged as follows:

IN =

$$\frac{1}{48}D_1 TR^2 D_r^2\phi_o\Big|_{R_1} - \frac{1}{48}D_2 TL^2 D_r^2\phi_o\Big|_{R_2} - \frac{1}{48}D_3 BL^2 D_r^2\phi_o\Big|_{R_3} + \frac{1}{48}D_4 BR^2 D_r^2\phi_o\Big|_{R_4}$$

$$\frac{1}{48}D_1 RT^2 D_{rz}\phi_o\Big|_{R_1} + \frac{1}{48}D_2 LT^2 D_{rz}\phi_o\Big|_{R_2} - \frac{1}{48}D_3 LB^2 D_{rz}\phi_o\Big|_{R_3}$$

$$- \frac{1}{48}D_4 RB^2 D_{rz}\phi_o\Big|_{R_4}$$

$$\frac{1}{48}D_1 r_i TR^2 D_{rzz}\phi_o\Big|_{R_1} - \frac{1}{48}D_2 r_i TL^2 D_{rzz}\phi_o\Big|_{R_2} - \frac{1}{48}D_3 r_i BL^2 D_{rzz}\phi_o\Big|_{R_3}$$

$$+ \frac{1}{48}D_4 r_i BR^2 D_{rzz}\phi_o\Big|_{R_4}.$$

$$\frac{1}{48}D_1 r_i RT^2 D_{rrz}\phi_o\Big|_{R_1} + \frac{1}{48}D_2 r_i LT^2 D_{rrz}\phi_o\Big|_{R_2} - \frac{1}{48}D_3 r_i LB^2 D_{rrz}\phi_o\Big|_{R_3}$$

$$- \frac{1}{48}D_4 r_i RB^2 D_{rrz}\phi_o\Big|_{R_4}$$

(C-11)

The evaluation of the derivatives, $D_r^2\phi_o\big|_{R_n}$, $D_{rz}\phi_o\big|_{R_n}$, $D_{rrz}\phi_o\big|_{R_n}$, and $D_{rzz}\phi_o\big|_{R_n}$ for n = 1,2,3,4, may be performed by using the flux expansions as cited in Appendix A and then applying the continuity of neutron current density between adjacent regions. Also, the terms that are of orders $O(h^4)$ and higher have to be lumped with the truncation error $O(h^4)$. Hence, the following formulations have been obtained:

a:

$$D_r^2\phi_o\big|_{R_1} = \frac{2}{D_1LR(R+L)}\,[LD_1(\phi_1 - \phi_o) + RD_2(\phi_3 - \phi_o)] + O(h) \tag{C-12}$$

$$D_r^2\phi_o\big|_{R_2} = \frac{2}{D_2LR(R+L)}\,[LD_1(\phi_1 - \phi_o) + RD_2(\phi_3 - \phi_o)] + O(h) \tag{C-13}$$

$$D_r^2\phi_o\big|_{R_3} = \frac{2}{D_3LR(R+L)}\,[LD_4(\phi_1 - \phi_o) + RD_3(\phi_3 - \phi_o)] + O(h) \tag{C-14}$$

$$D_r^2\phi_o\big|_{R_4} = \frac{2}{D_4LR(R+L)}\,[LD_4(\phi_1 - \phi_o) + RD_3(\phi_3 - \phi_o)] + O(h) \tag{C-15}$$

b:

$$D_{rz}\phi_o\big|_{R_1} = \frac{1}{2LRTD_1}\,[D_1L(\phi^1-\phi_1-\phi_2+\phi_o)-D_2R(\phi^2-\phi_3-\phi_2+\phi_o)] + O(h) \tag{C-16}$$

$$D_{rz}\phi_o\big|_{R_2} = \frac{1}{2LRTD_2}\,[D_1L(\phi^1-\phi_1-\phi_2+\phi_o)-D_2R(\phi^2-\phi_3-\phi_2+\phi_o)] + O(h) \tag{C-17}$$

$$D_{rz}\phi_o\Big|_{R_3} = \frac{1}{2RLBD_3}\,[D_3R(\phi^3-\phi_3-\phi_4+\phi_o)-D_4L(\phi^4-\phi_1-\phi_4+\phi_o)] + O(h) \tag{C-18}$$

$$D_{rz}\phi_o\Big|_{R_4} = \frac{1}{2RLBD_4}\,[D_3R(\phi^3-\phi_3-\phi_4+\phi_o)-D_4L(\phi^4-\phi_1-\phi_4+\phi_o)] + O(h) \tag{C-19}$$

c:

$$D_{rrz}\phi_o\Big|_{R_1} = \frac{2}{D_1(R+L)T}\left[\frac{D_1}{R}(\phi^1-\phi_1-\phi_2+\phi_o)+\frac{D_2}{L}(\phi^2-\phi_3-\phi_2+\phi_o)\right] + O(h) \tag{C-20}$$

$$D_{rrz}\phi_o\Big|_{R_2} = \frac{2}{D_2(R+L)T}\left[\frac{D_1}{R}(\phi^1-\phi_1-\phi_2+\phi_o)+\frac{D_2}{L}(\phi^2-\phi_3-\phi_2+\phi_o)\right] + O(h) \tag{C-21}$$

$$D_{rrz}\phi_o\Big|_{R_3} = -\frac{2}{D_3(L+R)B}\left[\frac{D_3}{L}(\phi^3-\phi_3-\phi_4+\phi_o)+\frac{D_4}{R}(\phi^4-\phi_1-\phi_4+\phi_o)\right] + O(h) \tag{C-22}$$

$$D_{rrz}\phi_o\Big|_{R_4} = -\frac{2}{D_4(L+R)B}\left[\frac{D_3}{L}(\phi^3-\phi_3-\phi_4+\phi_o)+\frac{D_4}{R}(\phi^4-\phi_1-\phi_4+\phi_o)\right] + O(h) \tag{C-23}$$

d:

$$D_{rzz}\phi_o\Big|_{R_1} = \frac{2}{D_1R(T+B)}\left[\frac{D_1}{T}(\phi^1-\phi_1-\phi_2+\phi_o)+\frac{D_4}{B}(\phi^4-\phi_1-\phi_4+\phi_o)\right] + O(h) \tag{C-24}$$

$$D_{rzz}\phi_o\Big|_{R_2} = \frac{-2}{D_2L(T+B)}\left[\frac{D_2}{T}(\phi^2-\phi_3-\phi_2+\phi_o)+\frac{D_3}{B}(\phi^3-\phi_3-\phi_4+\phi_o)\right] + O(h) \tag{C-25}$$

$$D_{rzz}\phi_o\Big|_{R_3} = \frac{-2}{D_3L(T+B)}\left[\frac{D_2}{T}(\phi^2-\phi_3-\phi_2+\phi_o)+\frac{D_3}{B}(\phi^3-\phi_3-\phi_4+\phi_o)\right] + O(h)$$

(C-26)

$$D_{rzz}\phi_o\Big|_{R_4} = \frac{2}{D_4R(T+B)}\left[\frac{D_1}{T}(\phi^1-\phi_1-\phi_2+\phi_o)+\frac{D_4}{B}(\phi^4-\phi_1-\phi_4+\phi_o)\right] + O(h)$$

(C-27)

2. Evaluation

$$\begin{aligned} IK = &- \frac{1}{24} D_1 r_i (R^3 D_{rrz}\phi_o + T^3 D_{rzz}\phi_o)\Big|_{R_1} \\ &- \frac{1}{24} D_2 r_i (L^3 D_{rrz}\phi_o - T^3 D_{rzz}\phi_o)\Big|_{R_2} \\ &+ \frac{1}{24} D_3 r_i (L^3 D_{rrz}\phi_o + B^3 D_{rzz}\phi_o)\Big|_{R_3} \\ &+ \frac{1}{24} D_4 r_i (R^3 D_{rrz}\phi_o - B^3 D_{rzz}\phi_o)\Big|_{R_4} \end{aligned}$$

(C-28)

The terms in Eq. C-28 may be arranged as

$$\begin{aligned} IK = &- \frac{1}{24} r_i R^3 \left[\frac{\partial^2}{\partial r^2}\left(D_1 \frac{\partial\phi_o}{\partial z}\Big|_{R_1} - D_4 \frac{\partial\phi_o}{\partial z}\Big|_{R_4}\right)\right] \\ &- \frac{1}{24} r_i L^3 \left[\frac{\partial^2}{\partial r^2}\left(D_2 \frac{\partial\phi_o}{\partial z}\Big|_{R_2} - D_3 \frac{\partial\phi_o}{\partial z}\Big|_{R_4}\right)\right] \\ &- \frac{1}{24} r_i T^3 \left[\frac{\partial^2}{\partial z^2}\left(D_1 \frac{\partial\phi_o}{\partial r}\Big|_{R_1} - D_2 \frac{\partial\phi_o}{\partial r}\Big|_{R_2}\right)\right] \\ &+ \frac{1}{24} r_i B^3 \left[\frac{\partial^2}{\partial z^2}\left(D_3 \frac{\partial\phi_o}{\partial r}\Big|_{R_3} - D_4 \frac{\partial\phi_o}{\partial r}\Big|_{R_4}\right)\right] \end{aligned}$$

(C-29)

By applying the continuity condition of the neutron current density, i.e., $D_1 \frac{\partial\phi_o}{\partial z}\Big|_{R_1} = D_4 \frac{\partial\phi_o}{\partial z}\Big|_{R_4}$, $D_2 \frac{\partial\phi_o}{\partial z}\Big|_{R_2} = D_3 \frac{\partial\phi_o}{\partial z}\Big|_{R_4}$, $D_1 \frac{\partial\phi_o}{\partial r}\Big|_{R_1} = D_2 \frac{\partial\phi_o}{\partial r}\Big|_{R_2}$, and $D_3 \frac{\partial\phi_o}{\partial r}\Big|_{R_3} = D_4 \frac{\partial\phi_o}{\partial r}\Big|_{R_4}$, one has

$$IK = 0 \qquad \text{(C-30)}$$

By substituting for $D_r^2\phi_o\Big|_{R_n}$, $D_{rz}\phi_o\Big|_{R_n}$, $D_{rrz}\phi_o\Big|_{R_n}$, and $D_{rzz}\phi_o\Big|_{R_n}$ in Eq. C-11 from the Eqs. C-12, C-13,---(C-26), and (C-27), respectively, and adding the result to Eq. C-30, one has

$$\begin{aligned} IN + IK = &- \frac{1}{48} D_1\sigma_1 \, r_i RT \, (\phi_1 + \phi_2 - 2\phi_o) \\ &- \frac{1}{48} D_2\sigma_2 \, r_i \, LT(\phi_2 + \phi_3 - 2\phi_o) \\ &- \frac{1}{48} D_3\sigma_3 \, r_i \, LB \, (\phi_4 + \phi_3 - 2\phi_o) \\ &- \frac{1}{48} D_4\sigma_4 \, r_i \, RB \, (\phi_4 + \phi_1 - 2\phi_o) \\ &+ \frac{1}{24} \frac{T}{LR} (R-1) [LD_1(\phi_1 - \phi_o) + RD_2(\phi_3 - \phi_o)] \\ &+ \frac{1}{24} \frac{B}{LR} (R-L) [LD_4(\phi_1 - \phi_o) + RD_3(\phi_3 - \phi_o)] \\ &+ \frac{1}{96} \frac{T}{LR}(L+R) [LD_1(\phi^1 - \phi_1 - \phi_2 + \phi_o) - RD_2(\phi^2 - \phi_3 - \phi_2 + \phi_o)] \end{aligned}$$

$$+ \frac{1}{96} \frac{B}{LR}(L+R) [LD_4(\phi^4-\phi_1-\phi_4+\phi_o)-RD_3(\phi^3-\phi_3-\phi_4+\phi_o)]$$

$$+ \frac{1}{24} \frac{r_i R}{TB}[D_1 B(\phi^1-\phi_1-\phi_2+\phi_o)+D_4 T(\phi^4-\phi_1-\phi_4+\phi_o)]$$

$$+ \frac{1}{24} \frac{r_i L}{TB}[D_2 B(\phi^2-\phi_3-\phi_2+\phi_o)+D_3 T(\phi^3-\phi_3-\phi_4+\phi_o)]$$

$$+ \frac{1}{24} \frac{r_i T}{LR}[D_1 L(\phi^1-\phi_1-\phi_2+\phi_o)+D_2 R(\phi^2-\phi_3-\phi_2+\phi_o)]$$

$$+ \frac{1}{24} \frac{r_i B}{LR}[D_3 R(\phi^3-\phi_3-\phi_4+\phi_o)+D_4 L(\phi^4-\phi_1-\phi_4+\phi_o)]$$

$$+ O(h^4) \qquad \text{(C-31)}$$

## 3. Evaluation

$$IM = - \frac{1}{16} \Sigma_1 r_i \; TR^2 D_r \phi_o \Big|_{R_1} - \frac{1}{16} \Sigma_1 r_i \; T^2 R \; D_z \phi_o \Big|_{R_1}$$

$$+ \frac{1}{16} \Sigma_2 r_i \; TL^2 \; D_r \phi_o \Big|_{R_2} - \frac{1}{16} \Sigma_2 r_i \; LT^2 \; D_z \phi_o \Big|_{R_2}$$

$$+ \frac{1}{16} \Sigma_3 r_i \; BL^2 \; D_r \phi_o \Big|_{R_3} + \frac{1}{16} \Sigma_3 r_i \; B^2 L \; D_z \phi_o \Big|_{R_3}$$

$$- \frac{1}{16} \Sigma_4 \; r_i BR^2 \; D_r \phi_o \Big|_{R_4} + \frac{1}{16} \Sigma_4 r_i \; B^2 R \; D_z \phi_o \Big|_{R_4}$$

(C-32)

The derivatives $D_r\phi_o\Big|_{R_n}$ and $D_z\phi_o\Big|_{R_n}$ can be evaluated using the flux expansion relations as in Appendix A and lumping all the terms of orders $O(h^4)$ and higher with the truncation error $O(h^4)$. The results may be written as follows

a: $$TR^2\, D_r\phi_o\Big|_{R_1} = TR(\phi_1 - \phi_o) + O(h^4) \quad \text{(C-33)}$$

b: $$T^2R\, D_z\phi_o\Big|_{R_1} = TR(\phi_2 - \phi_o) + O(h^4) \quad \text{(C-34)}$$

c: $$TL^2\, D_r\phi_o\Big|_{R_2} = -\, TL(\phi_3 - \phi_o) + O(h^4) \quad \text{(C-35)}$$

d: $$LT^2\, D_z\phi_o\Big|_{R_2} = TL(\phi_2 - \phi_o) + O(h^4) \quad \text{(C-36)}$$

e: $$L^2B\, D_r\phi_o\Big|_{R_3} = -\, LB(\phi_3 - \phi_o) + O(h^4) \quad \text{(C-37)}$$

f: $$-B^2L\, D_z\phi_o\Big|_{R_3} = BL(\phi_4 - \phi_o) + O(h^4) \quad \text{(C-38)}$$

g: $$BR^2\, D_r\phi_o\Big|_{R_4} = BR(\phi_1 - \phi_o) + O(h^4) \quad \text{(C-39)}$$

h: $$B^2R\, D_z\phi_o\Big|_{R_4} = -\, BR(\phi_4 - \phi_o) + O(h^4) \quad \text{(C-40)}$$

By substituting the Eqs. C-33, C-34,------, C-39, and C-40, respectively, into Eq. C-32, one has

$$IM = -\frac{1}{16}\,\Sigma_1\; r_i\; TR(\phi_1 + \phi_2 - 2\phi_o)$$

$$-\frac{1}{16}\,\Sigma_2\; r_i\; TL(\phi_3 + \phi_2 - 2\phi_o)$$

$$-\frac{1}{16}\,\Sigma_3\; r_i\; BL(\phi_3 + \phi_4 - 2\phi_o)$$

$$-\frac{1}{16}\,\Sigma_4\; r_i\; BR(\phi_1 + \phi_4 - 2\phi_o) \qquad \text{(C-41)}$$